# Science Fictions from the Old Classics in Physics

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## Old Classics

- Doppler Effect (1842).
- Wien’s displacement law (1893).
- de Broglie Wave or Matter Wave (1923).

## Science Fictions

- Doppler Shift of de Broglie waves.

- Equivalent form of Wien's displacement laws for Fermi gas and massive Bose gas.

Assumption: A Fermi gas or a massive Bose gas is replaced by an enormous number of de Broglie waves.

## Doppler Effect:

The pitch of an audible acoustic wave appears to change if there is a relative motion between the source and the observer. Such apparent change in frequency of sound waves is known as the Doppler effect in the case of acoustics.

Same kind of physical effect is also been observed in the case of optics with an apparent change in wavelength or color of the light.

With simple theoretical calculation one can show that the effect is common to all kinds of travelling waves.

However, to make appreciable change in frequency or wavelength for light waves the relative velocity must be a considerable fraction of the velocity of light.

Since the speed of light is extremely high compared to the speed of any terrestrial object, it is difficult to detect any measurable or observable change in wavelength or color of the emitted light from any source having relative motion with respect to the observer.

Whereas many of the stellar objects are moving with very high velocity, therefore wavelength of light emitted from these heavenly bodies, moving either towards or away from the earth show measurable amount of blue or redshift respectively in the spectral lines.

The possibility of shift in the position of a spectral line due to the relative motion of the source and the observer was first pointed out by Doppler in 1842.

## Wien's Displacement Law

A gas of Electromagnetic waves are enclosed in an enclosure at temperature $T$.

Wien's Displacement Law: $\lambda T =$Constant.

More acceptable form: $\lambda_m T =$Constant $\Longrightarrow$ Photon gas.

## Matter Waves or de Broglie Waves

Based on the theoretical explanation of Photo-Electric effect by Albert Einstein: Matter wave hypothesis was given by L. de Broglie in the year 1923.

Wavelength of matter wave or de Broglie wave:

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

## Wien's Displacement Laws For Fermi Gas and Massive Bose Gas

We consider a many body quantum system consisting of either fermions or massive bosons. Then without the loss of generality, we may assume that just like a black body system of electromagnetic waves or photon gas, the system is essentially a gas of an enormous number of de Broglie waves.

Work is based on three old classic pieces of discoveries— the Doppler effect in the year 1842, the Wien's displacement law in the year 1893 and the matter wave or the de Broglie wave in the year 1923.

Start with the Doppler Shift of de Broglie Waves associated with fermions or bosons.

The change in wavelength for de Broglie waves due to Doppler effect: The well known form of Lorentz transformations of particle momentum and energy:

$$p'_x = \gamma \left( p_x - \frac{VE}{c^2} \right) \quad \text{with} \quad \gamma = \left( 1 - \frac{V^2}{c^2} \right)^{-1/2} \quad \text{and} \quad p'_y = p_y$$

$$E' = \gamma (E - V p_x)$$

We assume two frame of references: $S \longrightarrow$ the rest frame, $S' \longrightarrow$ moving with respect to $S$ with a uniform velocity $V$ along $x$-direction. We further assume that the motion of the particle is on $x - y$ plane.

Now from the definition of de Broglie wavelengths:

$$\frac{\lambda}{\lambda'} \cos\theta' = \gamma \left[ \cos\theta - \frac{V}{c} \left\{ 1 + \left( \frac{\lambda}{\lambda_c} \right)^2 \right\}^{1/2} \right]$$

$E = (p^2 c^2 + E_0^2)^{1/2} \longrightarrow$ the particle energy in $S$ frame and a similar expression for $E'$ in $S'$ frame, with $E_0 = m_0 c^2 \longrightarrow$ the rest mass energy and $\lambda_c = h/m_0 c \longrightarrow$ the Compton wavelength.

Aberration is given by:

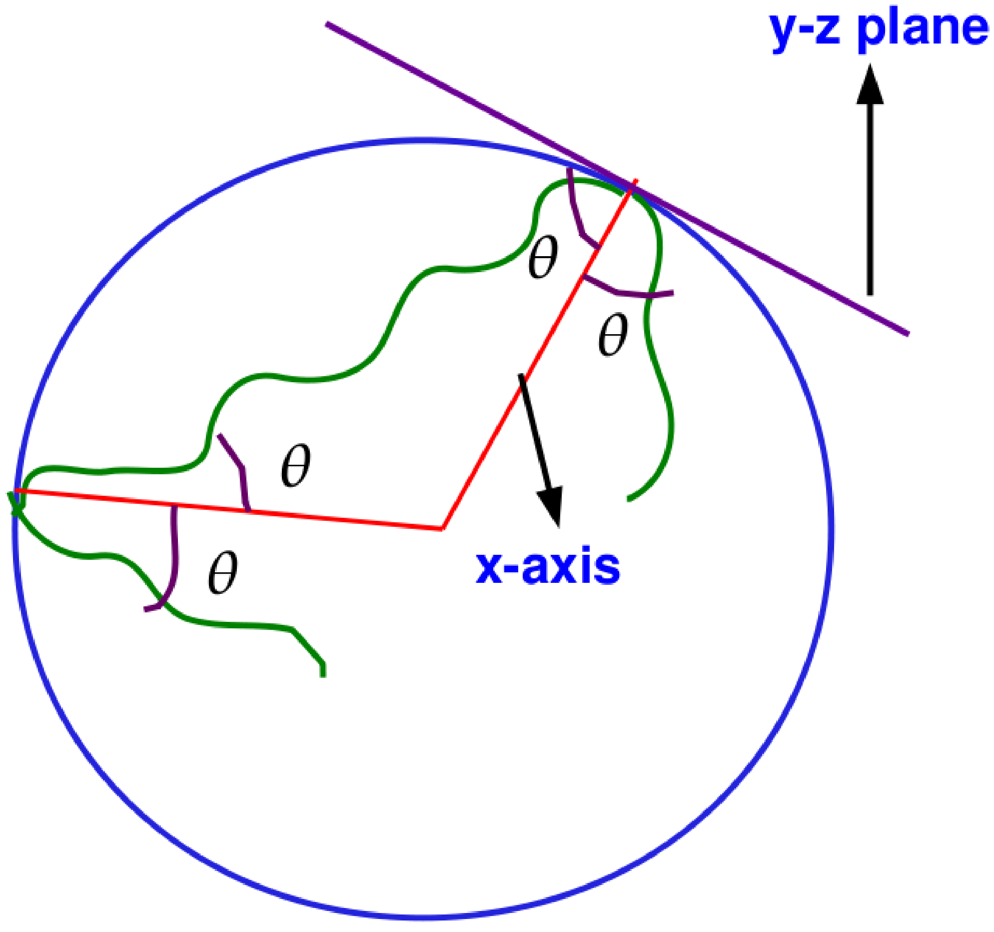
$$\tan\theta' = \frac{\sin\theta}{\gamma \left[ \cos\theta - \frac{V}{c} \left\{ 1 + \left( \frac{\lambda}{\lambda_c} \right)^2 \right\}^{1/2} \right]}$$

To obtain the Doppler shift of the de Broglie waves for the particles it is more convenient to start from the Lorentz transformation for the particle energy, Then:

$$\frac{\lambda}{\lambda'} = \gamma\left[1 - \frac{2V}{c}\cos\theta\left\{1 + \left(\frac{\lambda}{\lambda_c}\right)^2\right\}^{1/2} + \left(\frac{V}{c}\right)^2\left\{\cos^2\theta + \left(\frac{\lambda}{\lambda_c}\right)^2\right\}\right]^{1/2}$$

## Variation of de Broglie Waves with Temperature:

We consider either a Fermi gas or a gas of massive bosons in an enclosure. For the sake of simplicity the enclosure is assumed to be spherical in nature, wall is a perfect reflector and is slowly moving outward with a velocity $V$, where $V \ll c$.


y-z plane
θ
θ
θ
θ
x-axis

The moving wall $\longrightarrow S'$ frame, whereas the $S$ frame is at rest inside the enclosure with a fictitious observer sitting there.

The tangential plane at some arbitrary point on the outer surface of the wall is assumed to be in the $y - z$ plane.

Then the normal drawn from the centre to this point of intersection is the $x$-direction.

The particle which is hitting the wall at this point of intersection is as before assumed to be moving in $x - y$ plane.

Then for a de Broglie wave of wavelength $\lambda$, the point of intersection on the moving wall at which it is hitting is equivalent to an observer moving away from the source, which is radially outward along $x$-direction.

As a consequence the received de Broglie wave at the moving wall will be red shifted and is given by

$$\frac{\lambda}{\lambda'} = \left[1 - \frac{2V}{c}\frac{\lambda}{\lambda_c}\cos\theta\right]^{1/2}$$

where we have neglected the term $(V/c)^2$.

When the particle is reflected back from the point of incidence, since the wall is moving outward, it is equivalent to the emission from a source moving away from the observer.

Therefore in this case also the de Broglie wave of the particle will be red shifted as observed from $S$ frame.

Combining these two effects, the relation between the final red shifted wavelength to that of the original one is given by

$$\frac{\lambda''}{\lambda} = \frac{\left[1 + \frac{2V}{c}\frac{\lambda''}{\lambda_c}\cos\theta\right]^{1/2}}{\left[1 - \frac{2V}{c}\frac{\lambda}{\lambda_c}\cos\theta\right]^{1/2}}$$

Since $V/c \ll 1$, we have approximately

$$\frac{\lambda''}{\lambda} \approx \frac{\left[1 + \frac{V}{c}\frac{\lambda}{\lambda_c}\cos\theta\right]}{\left[1 - \frac{V}{c}\frac{\lambda}{\lambda_c}\cos\theta\right]}$$

Assuming that the amount of final red shift is infinitesimal, the above relation may further be approximated to

$$\frac{\lambda''}{\lambda} \approx 1 + \frac{2V}{c}\frac{\lambda}{\lambda_c}\cos\theta$$

Writing the final red shifted wave length $\lambda'' = \lambda + d\lambda$, we have the resultant infinitesimal change in wave length

$$d\lambda = \frac{2V}{c}\frac{\lambda^2}{\lambda_c}\cos\theta$$

To eliminate the arbitrary angle of incidence $\theta$, we consider multiple reflection of de Broglie waves from the inner wall of the enclosure. For spherical geometrical structure, with radius $r$, a de Broglie wave travels a distance $2r\cos\theta$ between two successive collisions. Therefore the number of reflections per unit time is $v/2r\cos\theta$. Hence the change in wavelength per unit time is

$$d\lambda = \frac{V}{c}\frac{\lambda^2}{\lambda_c}\frac{dr}{r}$$

where we have assumed that the particle travels $\delta r$ distance in unit time and in the limiting case it is $dr$.

First law of thermodynamics for an adiabatic change $\longrightarrow dQ = dU + PdV_0 = 0$.

$\Longrightarrow$ for a non-relativistic Fermi or Bose gas $PV_0^{5/3}$ =constant.

From standard results $\Longrightarrow$ energy density for free Fermi gas or Bose gas:

$$\epsilon = \frac{3}{2}\frac{kT}{\lambda_0^3} f_{5/2}(z) \text{ and } \epsilon = \frac{3}{2}\frac{kT}{\lambda_0^3} g_{5/2}(z) \text{ respectively,}$$

where
$$f_{5/2}(z) = \sum_{l=1}^{\infty} (-1)^{l-1} \frac{z^l}{l^{5/2}}, \;\; g_{5/2}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}}, \;\; \lambda_0 = \frac{\hbar}{(2\pi mkT)^{1/2}}$$

and $z = \exp\left(\frac{\mu}{kT}\right)$

the fugacity, $\mu \longrightarrow$ chemical potential of the constituents.

$f_\nu(z) \longrightarrow$ Fermi function and $g_\nu(z) \longrightarrow$ Bose function.

Many body Fermi system $\longrightarrow$electron gas in a piece of metal or inside white dwarfs or neutron matter inside neutron stars $\longrightarrow \mu \neq 0 \Longrightarrow$an analytical expression for energy density for a Fermi gas can not be obtained.

Fermi system $\longrightarrow$ Boltzmann statistics. $\Longrightarrow$

$$P = \frac{2}{3}\epsilon \;\text{ and }\; P \propto \exp\left(\frac{\mu}{kT}\right) T^{5/2}$$

$\Longrightarrow$

$$\exp\left(\frac{2\mu}{5kT}\right) T r^2 = \text{ constant}$$

where $V_0 = 4\pi r^3/3 \longrightarrow$ volume of the enclosure.

Taking log of both the sides and then differentiating all the terms $\Longrightarrow$

$$\left(-\frac{2\mu}{5k}\frac{1}{T^2} + \frac{1}{T}\right) dT = -2\frac{dr}{r} = -2\frac{c}{V}\frac{\lambda_c}{\lambda}\frac{d\lambda}{\lambda}$$

Integrating, rearranging the terms and finally redefining $\lambda$ and $T$ as $\lambda^*$ and $T^*$, we have

$$\frac{1}{\lambda^*} - \frac{1}{T^*} = \ln\left(\frac{T}{T_0}\right)$$

where

$$\frac{1}{\lambda^*} = \frac{2\lambda_c c}{V}\frac{1}{\lambda}, \quad \frac{1}{T^*} = \frac{2\mu}{5k}\frac{1}{T}$$ and $T_0$ is a positive constant

$\Longrightarrow$ equation for a thin lens $\longrightarrow$ right hand side is the inverse of focal length.

At high temperature a Fermi gas $\longrightarrow$Boltzmann gas $\Longrightarrow$ de Broglie wavelengths $\longrightarrow \infty$. $\Longrightarrow$ right hand side must be negative $\Longrightarrow$ $T < T_0$ $\Longrightarrow$ equation for the convex lens.

Numerical solution $T_{root}$ of the equation $T^* \ln(T/T_0) + 1 = 0$ $\Longrightarrow$ the temperature at which a fermion behaves like a classical particle.

Above this temperature lens like equation does not hold.

As $T \longrightarrow 0$ $\Longrightarrow$the second term on the left hand side of lens like equation $\longrightarrow$ infinitely large much before the right hand side goes to $-\infty$.

**Conclusions:** As temperature decreases, the de Broglie wavelength also decreases $\Longrightarrow$ the quantum mechanical effect becomes more and more important.

**Extreme situation:** $T \longrightarrow 0 \Longrightarrow \lambda \longrightarrow 0$.

**Comparison with convex lens:** $\lambda = \infty \longrightarrow$ focal plane $\longrightarrow$ real image at infinity.

**Temperature at which $\lambda = \infty$ separating the temperature space into a quantum zone and a classical zone.**

Beyond the focal plane away from the convex lens $\longrightarrow$ images are always real.

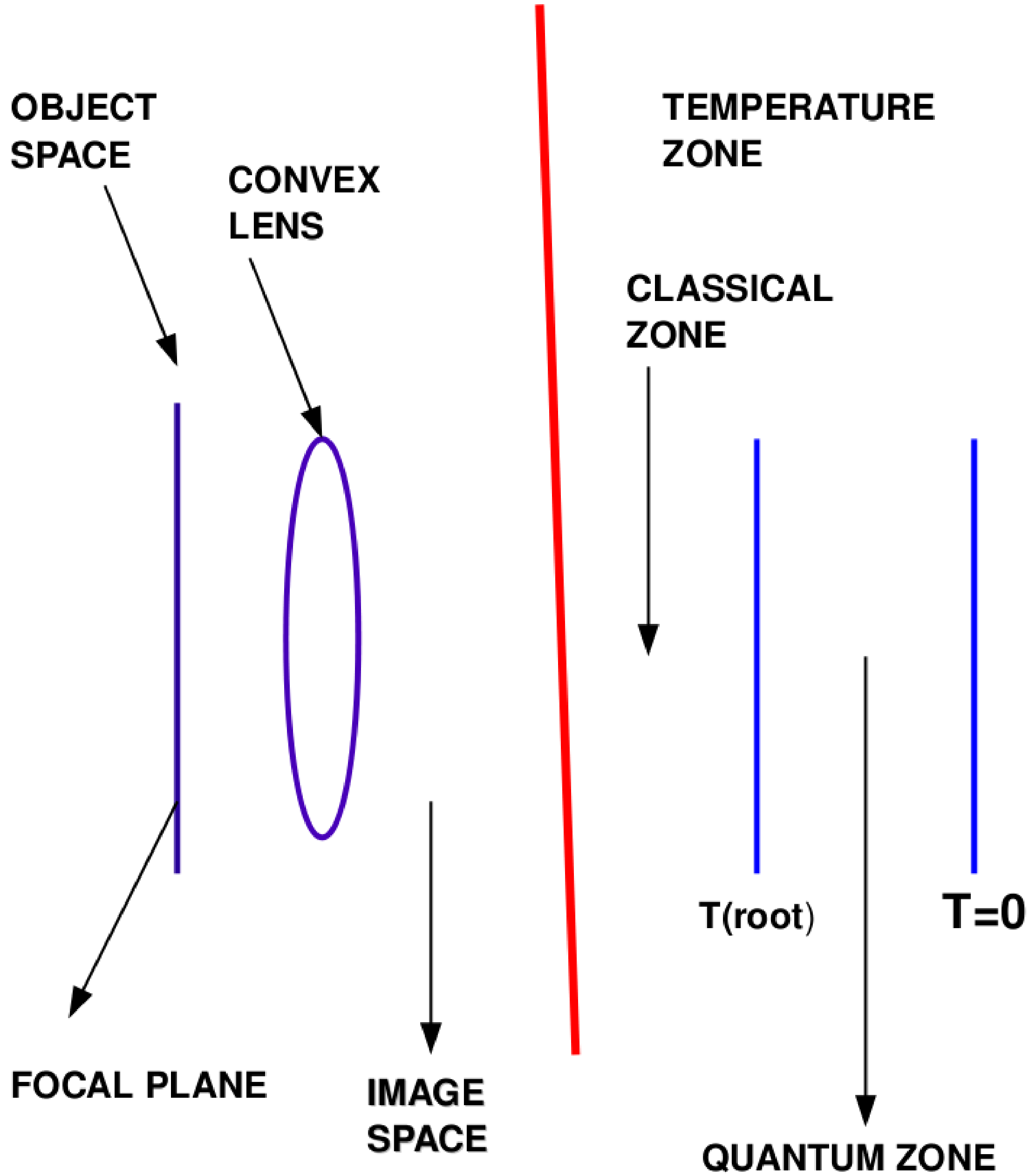
In the present scenario: The system becomes classical.

In the case of convex lens the object space between the focal plane and the lens always produce virtual images.

Same kind of picture is true here also.

The temperature zone between the upper critical value and $T \longrightarrow 0$ is the quantum mechanical region.

So the quantum mechanical region of temperature in the present scenario corresponds to the object space producing virtual images in the case of a convex lens.


OBJECT SPACE
CONVEX LENS
TEMPERATURE ZONE
CLASSICAL ZONE
T(root)
T=0
FOCAL PLANE
IMAGE SPACE
QUANTUM ZONE

In the quantum mechanical zone, because of uncertainty principle the exact location of the particle can not be predicted, only the probability of existence at a point can be obtained from the wave function of the particle.

Therefore grossly speaking, a cloudy picture will be observed instead of a real location of the particle.

Bose system: $\longrightarrow \pi^{+} - \pi^{-}$ matter or neutral pion matter or even a system of extremely rarefied cold atoms.

For bosonic systems there is as such no conserved quantum number $\Longrightarrow \mu = 0$ for the constituents.

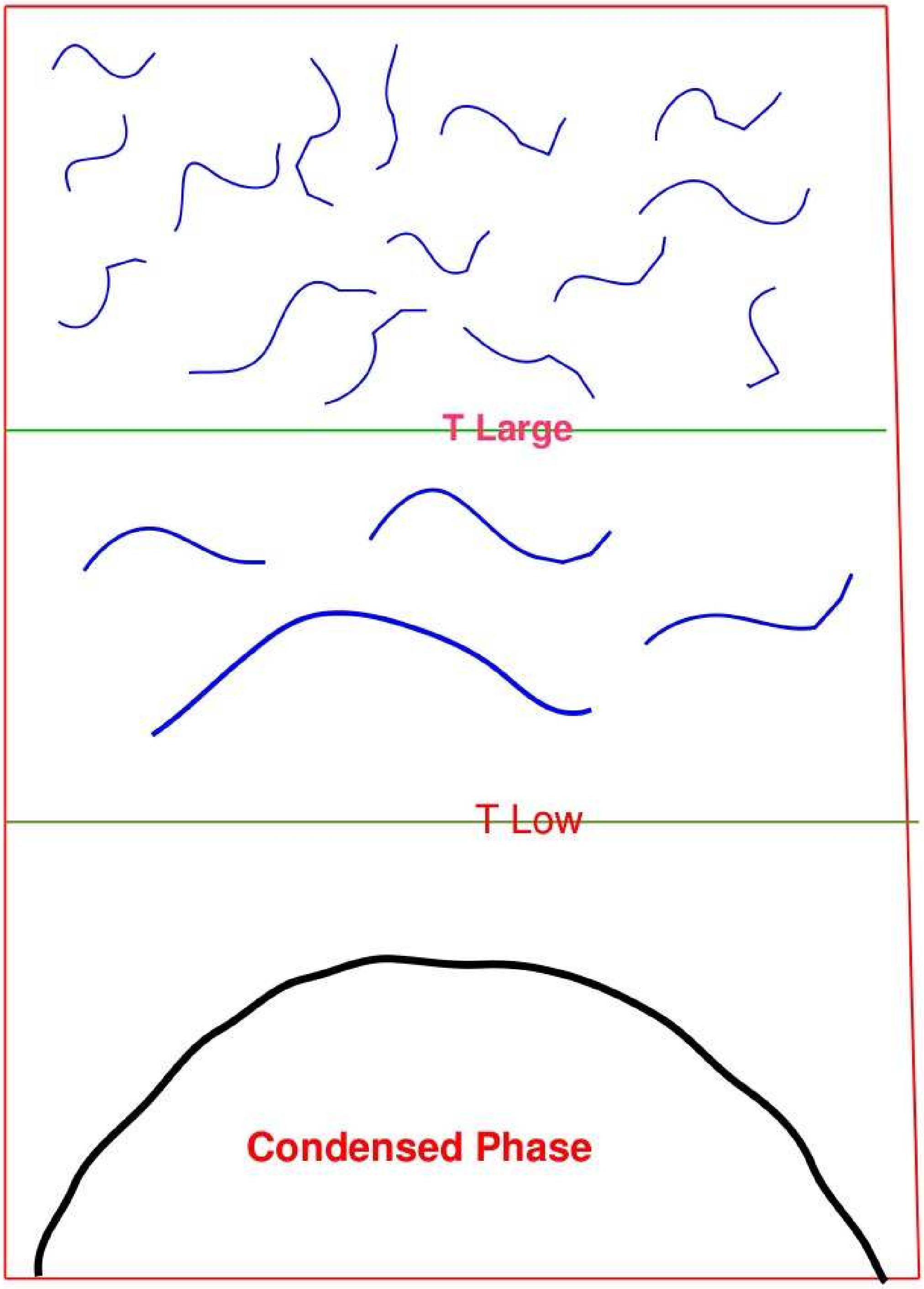
The series $g_{5/2}(1)$ can be expressed in terms of known Zeta-function

$$\frac{1}{\lambda} = \frac{V}{2\lambda_c c} \ln\left(\frac{T}{T_0}\right) \quad \text{or} \quad \lambda \ln\left(\frac{T}{T_0}\right) = \text{constant}$$

$T_0$ is the minimum value of temperature for a Bose system at which $\lambda = \infty$.

$T_0$ Bose condensation temperature of the gas.

$T_0$ for Bose gas carrying quite different physical meaning.


T Large
T Low
Condensed Phase

Since in the condensed phase all the bosons occupy the same quantum state, the spatial coherence length will be large enough in the atomic scale.

In this simple model it is reflected by the extremely large value of de Broglie wavelength, which is large enough in the quantum scale.

Then as the temperature increases, the quantum effect dominates more and more and the system becomes incoherent because of randomness. This is just opposite in nature compared to Fermi gas.

For mass-less bosons, i.e. with $m = 0$, since $\lambda_c = \infty$, the above equation can not predict the condensation temperature. Which is already known for photon gas and phonon gas.

**Final Remark:** It is therefore quite surprising that based on three very old classic pieces of discoveries- the Doppler effect in the year 1842, the Wien's displacement law in the year 1893 and the Matter wave or the de Broglie wave in the year 1923, it is possible to obtain the variation of de Broglie wavelength with temperature for fermions and bosons in a many particle system.

It is also possible to obtain the temperature beyond which the fermionic system behaves classically, the critical temperature for Bose condensation and also one can conclude that for mass-less bosons the critical temperature of Bose condensation can not be predicted.

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