# A THEORETICAL STUDY OF THE EFFECT OF STRONG ELECTRIC AND MAGNETIC FIELDS ON COMPACT STELLAR OBJECTS - AN APPLICATION TO TINY LABORATORY SAMPLES


Arpita Ghosh

Department of Physics, Visva-Bharati


For

Pre-Ph.D. Seminar, Department of Physics

Visva-Bharati

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# CONTENTS

| Objects | Strength (Gauss) |
|---|---|
| Earth | 0.6 |
| Bar magnet | 100 |
| Strongest steady field in the laboratory | $4 \times 10^5$ |
| Strongest man made field (millisecond duration) | $10^7$ |
| Maximum field for ordinary star | $10^6$ |
| Typical field for a radio pulsar | $10^{12}$ |
| Millisecond pulsars | $10^8$ |
| Magnetars | $10^{14} - 10^{15}$ |

**Brief Overview of Magnetic Field Strength in Natural System**

## Effect of strong magnetic field on crustal matter of magnetars

- Capable of distorting and magnifying images of stellar objects (magnetic lensing).

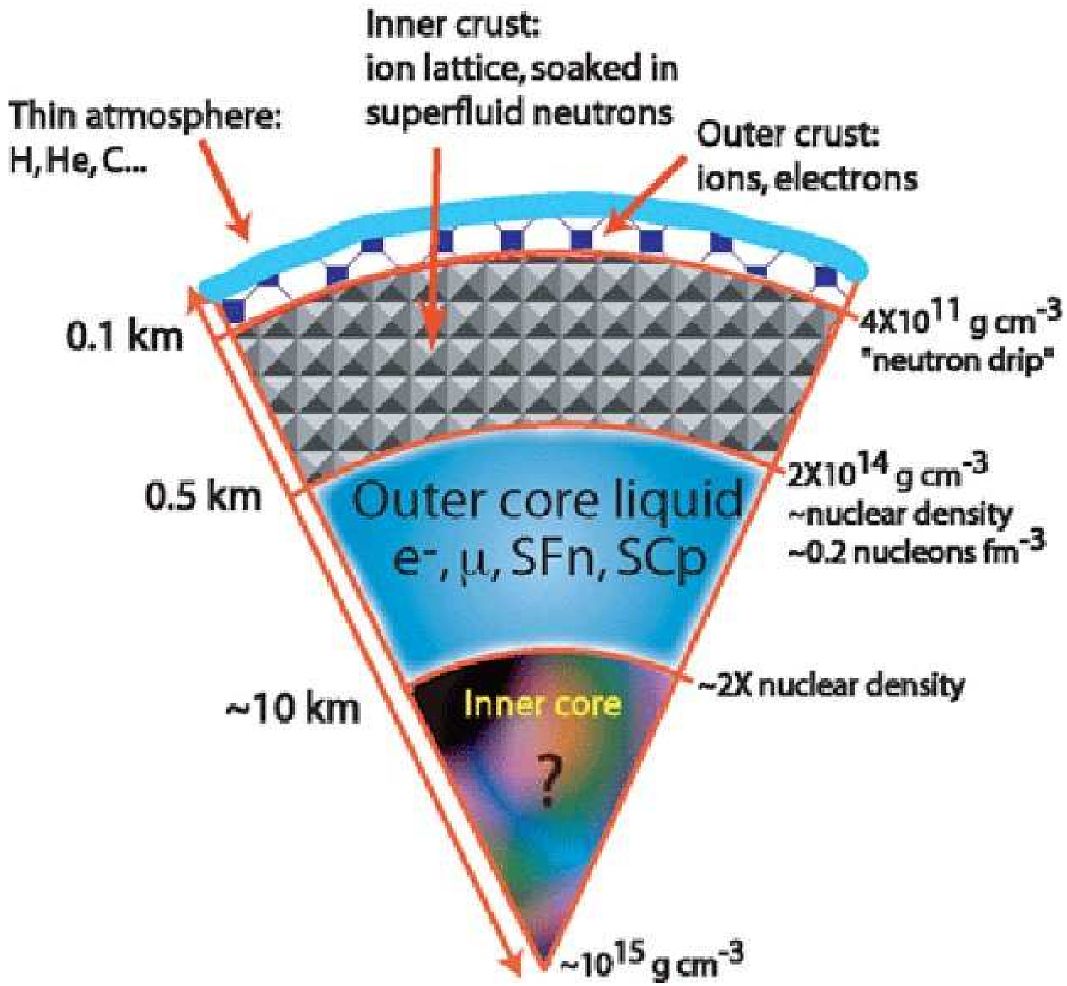
- Change of self-energy of electrons.

- Cause photons to rapidly split and merge.

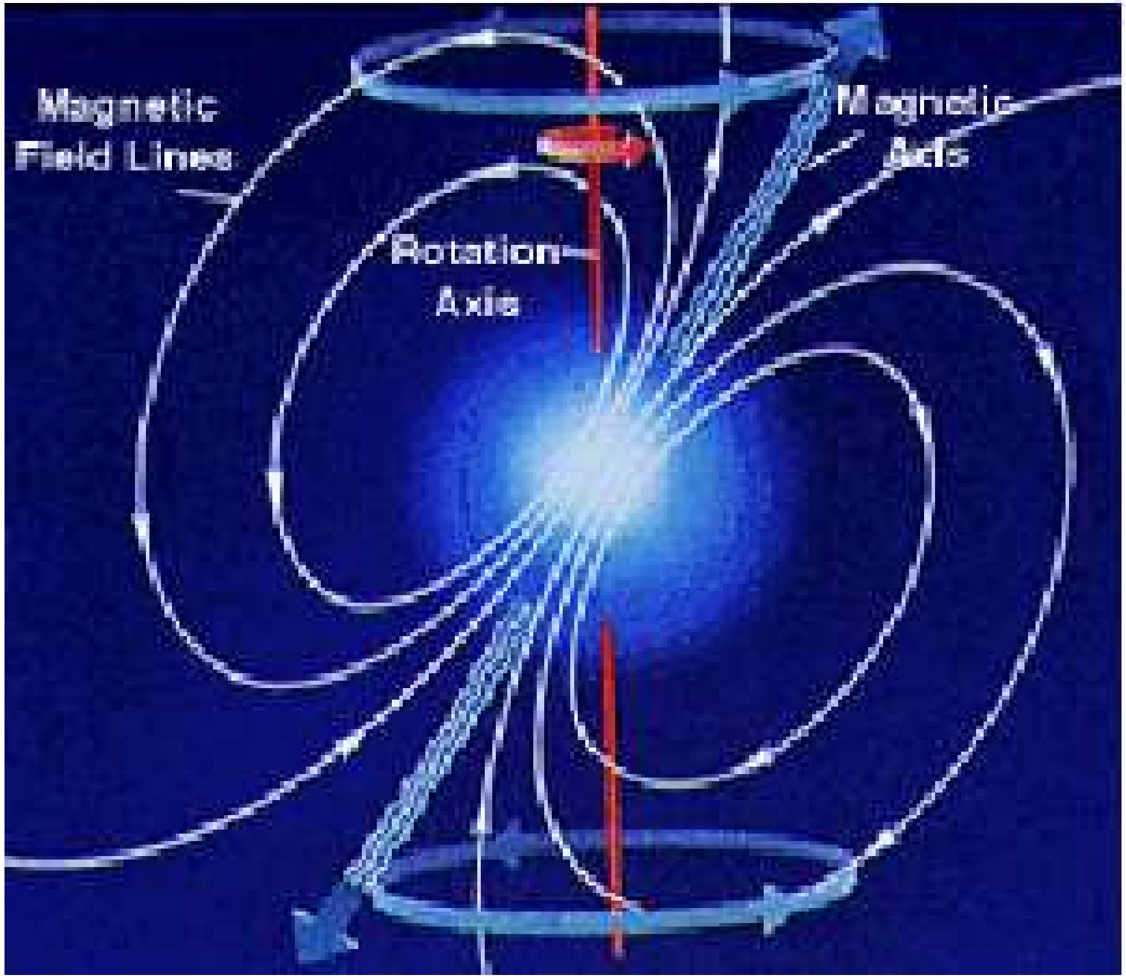
- **Distort atoms into long thin cylindrical shape**.

- Affects equation of state for dense stellar matter.
- Affects elementary process (electromagnetic and weak).
- **Affects electron emission process from polar region of magnetars.**
- **Work function of crustal matter associated with electron emission becomes anisotropic.**

## Schematic Diagram for the Internal Structure of a Typical Neutron Star/Magnetar



## Typical Structure of a Rotating Neutron Star/Magnetar with Magnetic Lines of Forces



## Inner crust matter of Magnetars

- Surface magnetic field of magnetars: $\geq 10^{14}$ G

- Inner crust region: Width $\sim 0.5 - 0.7$ km, Density $\sim 10^{11} - 10^{14}$ gm/cc

- Structure of inner crust :

  Array of cylindrically deformed (cigar shaped) metallic ions (mainly metallic iron).
  $\Longrightarrow$
  Cylindrical type distribution of electron gas around each nucleus $\longrightarrow$
  axis of each cylinder along the direction of magnetic field $\longrightarrow$
  we assume azimuthal symmetry for electron distribution.

## Quantum Nature of Electron Gas

- Cylindrical distribution of electrons around the nucleus $\longrightarrow$ distorted Wigner-Seitz (WS) cells.

- Landau levels of these electrons are populated.

- Proton are massive enough $\Longrightarrow$ no quantum mechanical effect of magnetic field.

## Formalism in brief

- Relativistic version of Thomas-Fermi method in presence of strong magnetic field.
- Magnetic field $B$ is constant and along $z$-axis ($A^{\mu} = 0, 0, xB, 0$).

- We start with Poisson equation:

$$\nabla^2\phi = 4\pi e n_e(r - r_n) - \frac{4\pi Z e}{V_n}\theta(r_n - r) \tag{1}$$

- In cylindrical coordinate the contribution of protons $\longrightarrow$ $\theta(r_n - r)\theta(z_n - z)$

- Density of degenerate electron gas :

$$n_e = \frac{eB}{2\pi^2}\sum_{\nu=0}^{\nu_{max}}(2 - \delta_{\nu 0})p_F, \; \nu \rightarrow \text{ Landau QN}, \; \nu_{max} \rightarrow \text{ upper limit} \tag{2}$$

- To solve the Poisson equation in cylindrical coordinate we use Thomas-Fermi condition :

$$\mu_e = (p_F^2 + m_e^2 + 2\nu eB)^{1/2} - e\phi = \text{const.} \tag{3}$$

- In strong magnetic field we get a magnetic constriction in the direction perpendicular to the field, and the electron distribution around the nucleus becomes cigar shaped.

- Here for the sake of simplicity, we consider cylindrical type shape.

- Radial contraction factor:

$$\gamma \propto 4\left(\frac{a_0}{\rho}\right)^2 \propto \frac{B}{B_0}, \tag{4}$$

  where B is external magnetic field, $B_0 \approx 10^8$G, $a_0$ is the Bohr radius and $\rho$ is the radial parameter.

- We start with cylindrical form of Poisson equation:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = \lambda^2\psi \tag{5}$$

Where $\lambda = 2e^3B/\pi$.

- With the separation of variables $\psi(r,z) = R(r)L(z)$:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \xi^2 R = 0, \tag{6}$$

$$\frac{d^2L}{dz^2} - (\xi^2 + \lambda^2)L = 0, \tag{7}$$

- The combined solution:

$$\psi(r,z) = CJ_0(\xi r)\exp\left[\pm(\xi^2+\lambda^2)^{1/2}z\right] \tag{8}$$

where $\xi(B)$ is a real constant and $C(B)$ is normalization constant.

- Unlike spherical distribution of electrons around the nucleus; in the cylindrically distorted case surface electric field can not be zero.

- Electric field on the curved surface $E_1$

$$E_1 = -2C\,\mathrm{exp}(-\Lambda(s^2 - r_{max}^2)^{1/2})(J_0(\xi r_{max})\Lambda\hat{e_z} + J_1(\xi r_{max})\xi\hat{e_r}) \quad (9)$$

- similarly at the plane faces $E_2$

$$\begin{aligned} E_2 &= -2C\,\mathrm{exp}(-\Lambda z_{max})(J_0(\xi(s^2 - z_{max}^2)^{1/2})\Lambda\hat{e_z} \\ &+ J_1(\xi(s^2 - z_{max}^2)^{1/2})\xi\hat{e_r}) \end{aligned} \quad (10)$$

where $\Lambda = (\lambda^2 + \xi^2)^{1/2}$

- Due to electromagnetic induction there will be charge polarization on the surfaces of cylindrically deformed WS cells.

- As a result $\approx 100$ charged WS cells are observed to form a bundle of charge neutral combination, instead of a single cylindrical cell.

- The electron Fermi momentum, $p_F(r,z)$:

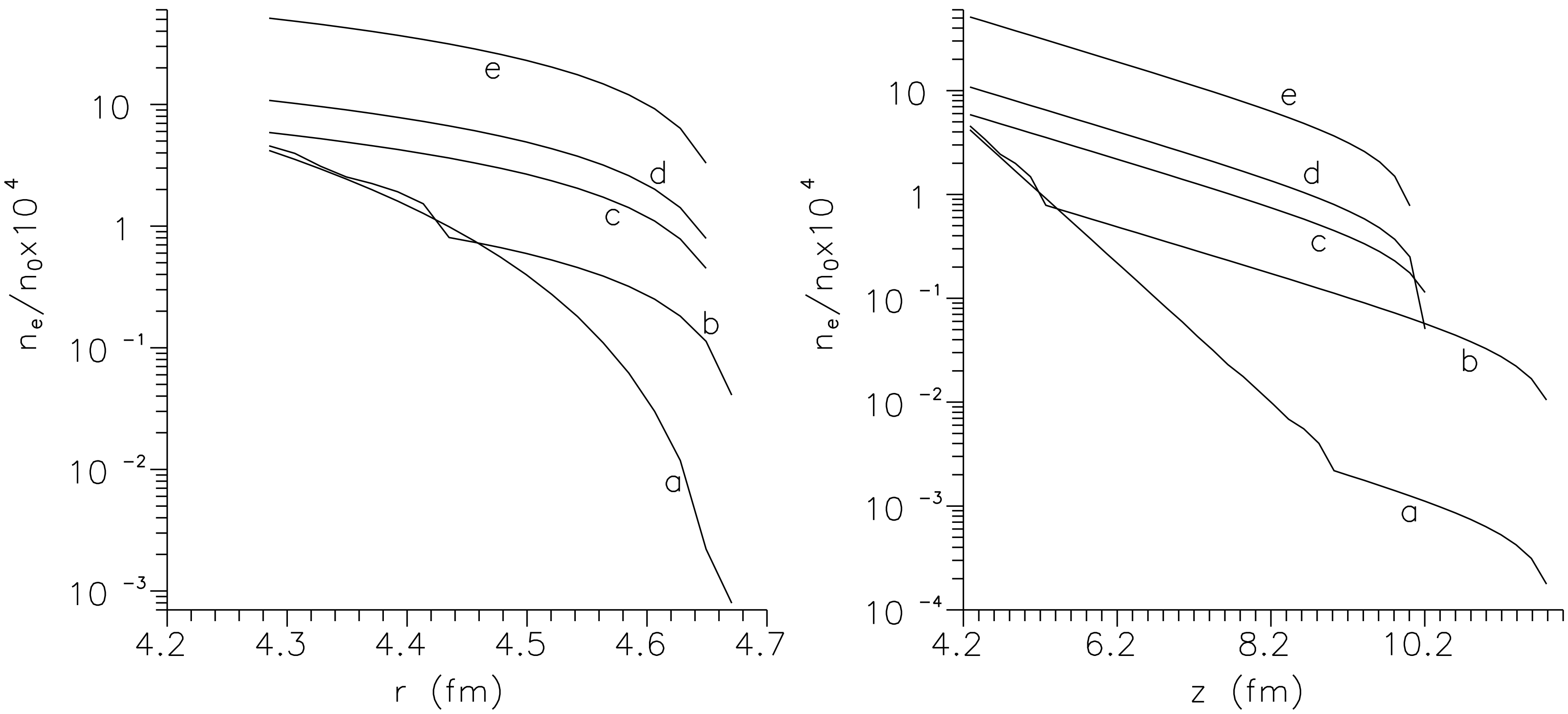
$$p_F = \left[\psi^2(r,z) - m_\nu^2\right]^{1/2}, \text{ with } m_\nu = (m^2 + 2\nu eB)^{1/2} \tag{11}$$

- Electron density:

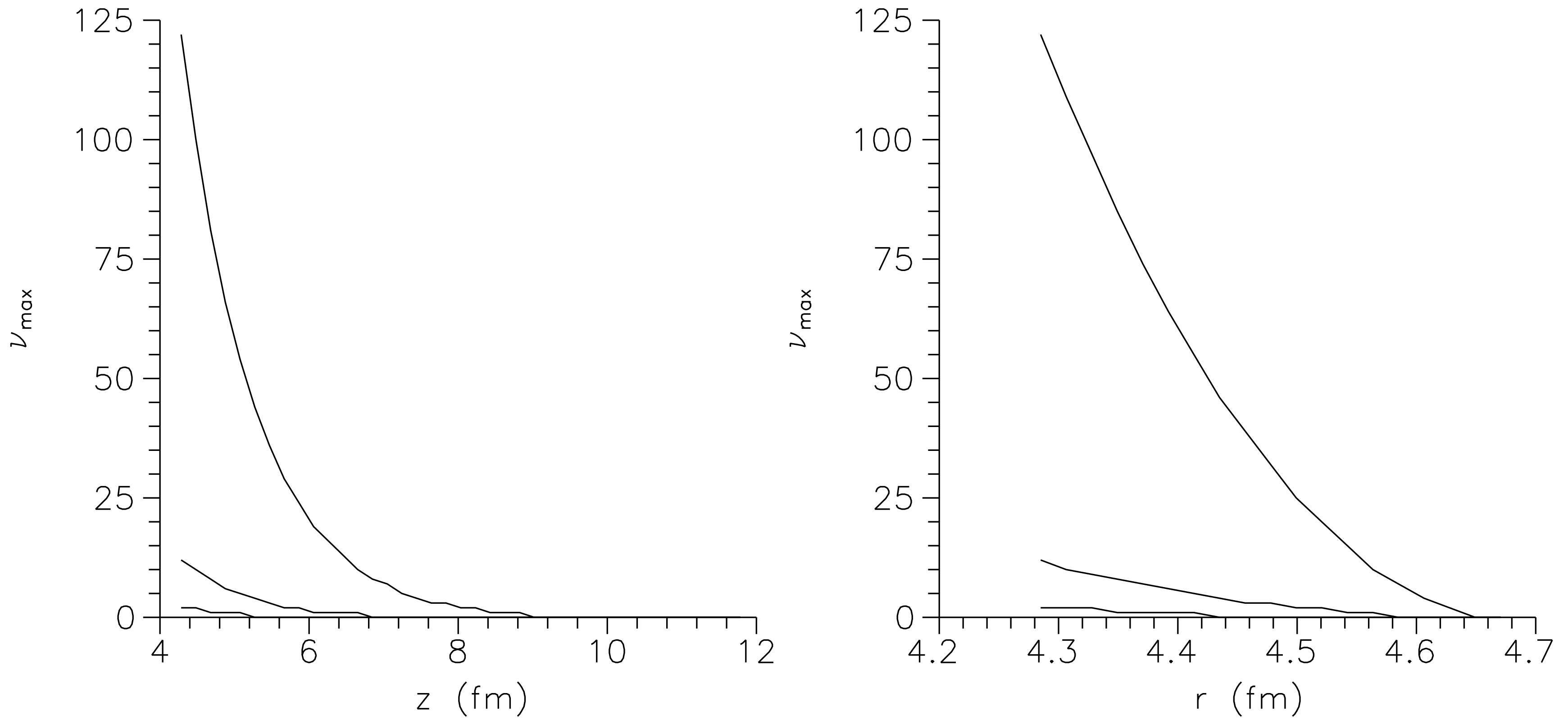
$$n_e(r,z) = \frac{eB}{2\pi^2} \sum_{\nu=0}^{\nu_{max}} (2 - \delta_{\nu 0}) \left[\psi^2(r,z) - m_\nu^2\right]^{1/2} \tag{12}$$

- The upper limit of Landau quantum $\nu$:

$$\nu_{max} = \frac{\psi^2(r,z) - m_e^2}{2eB} = \nu_{max}(r,z) \tag{13}$$



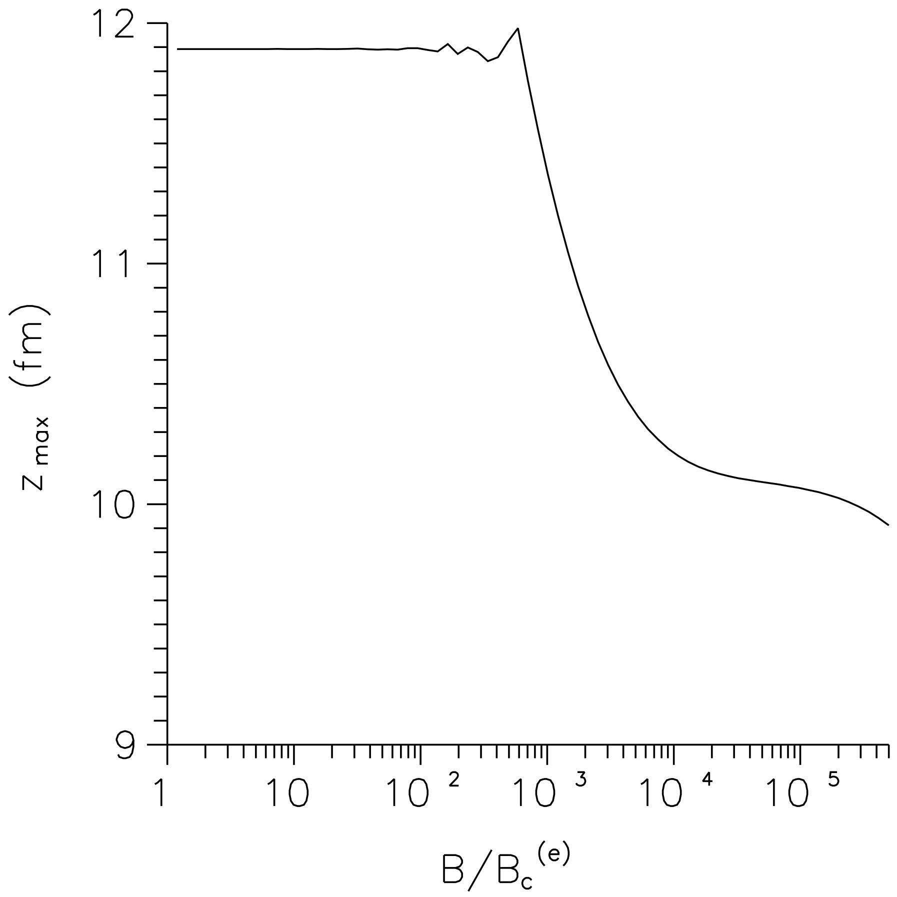

Variation of $n_e$ with $r$ and $z$

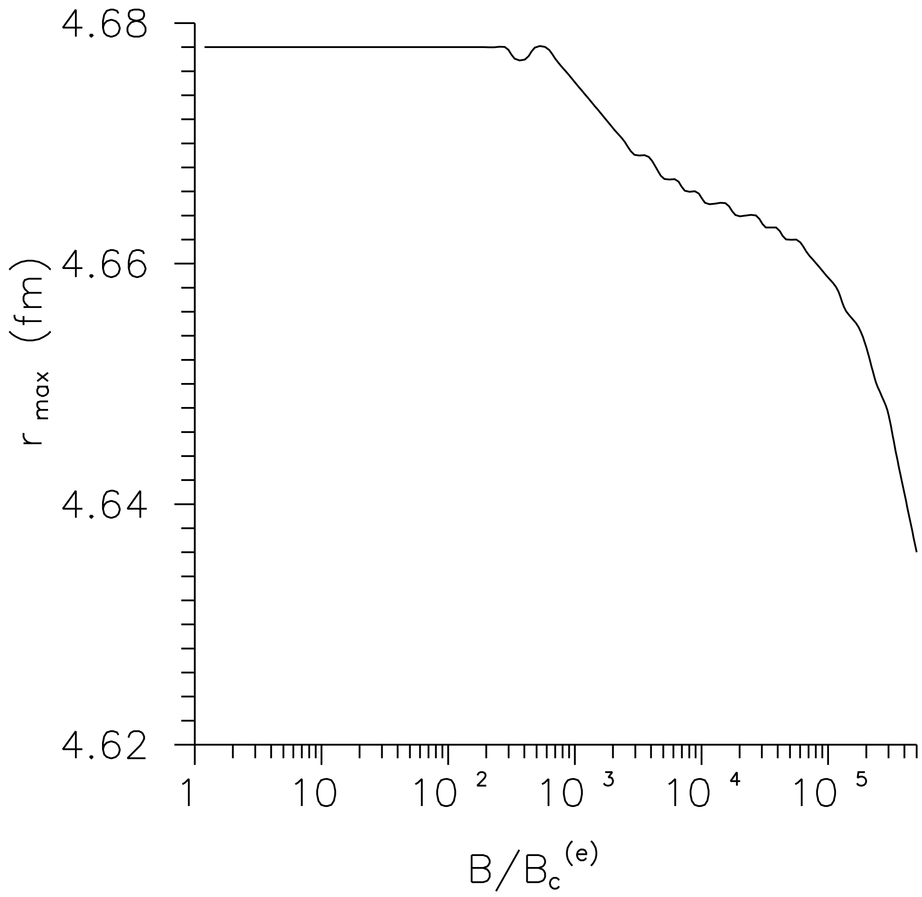



Variation of $\nu_{max}$ with $z$ and $r$

Variation of $z_{max}$ and $r_{max}$ with B :

- From $\nu_{max}(r_{max}, z_{max}) = 0$



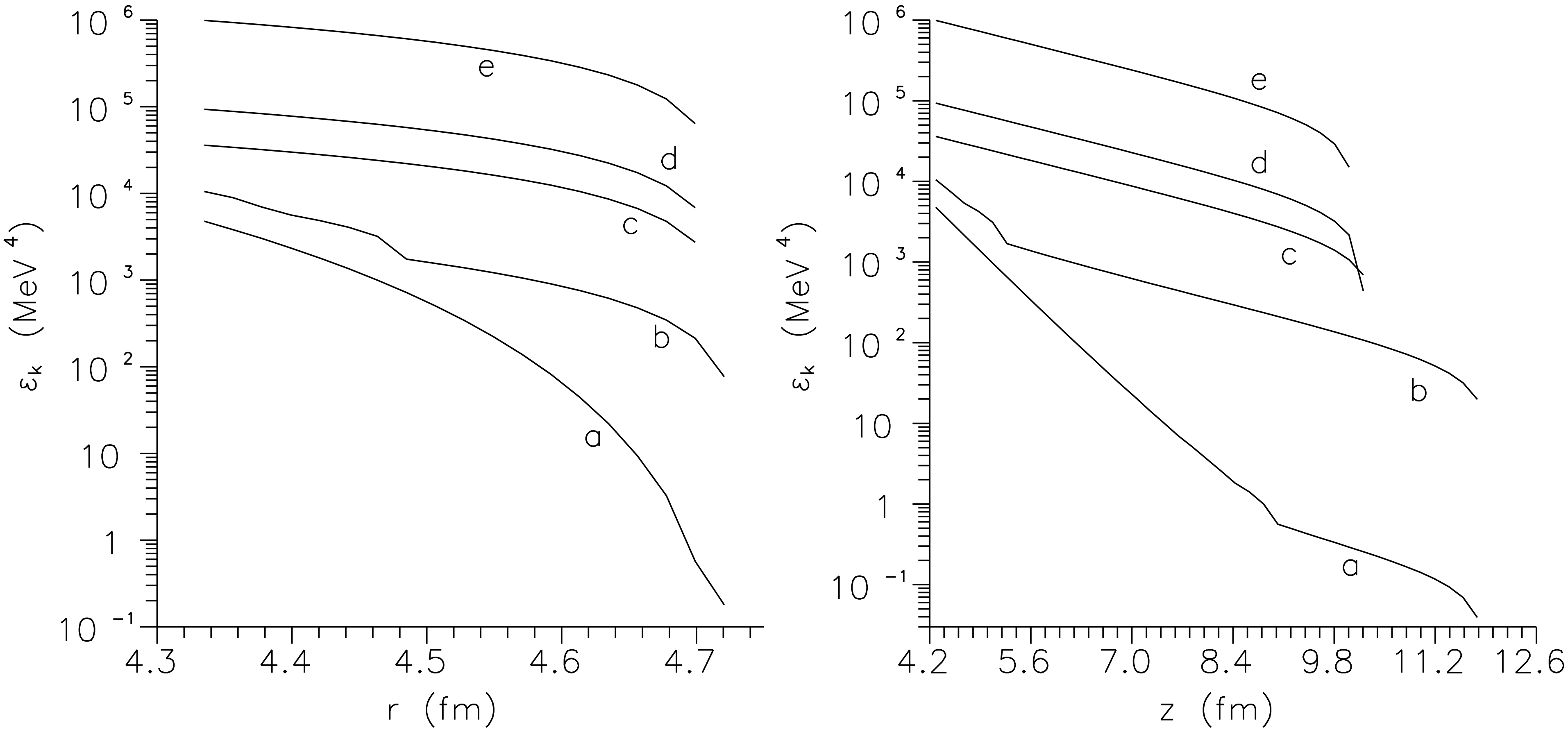
Different kinds of energies associated with the electron gas within WS cells –

ⓐ Cell-averaged kinetic energy density :

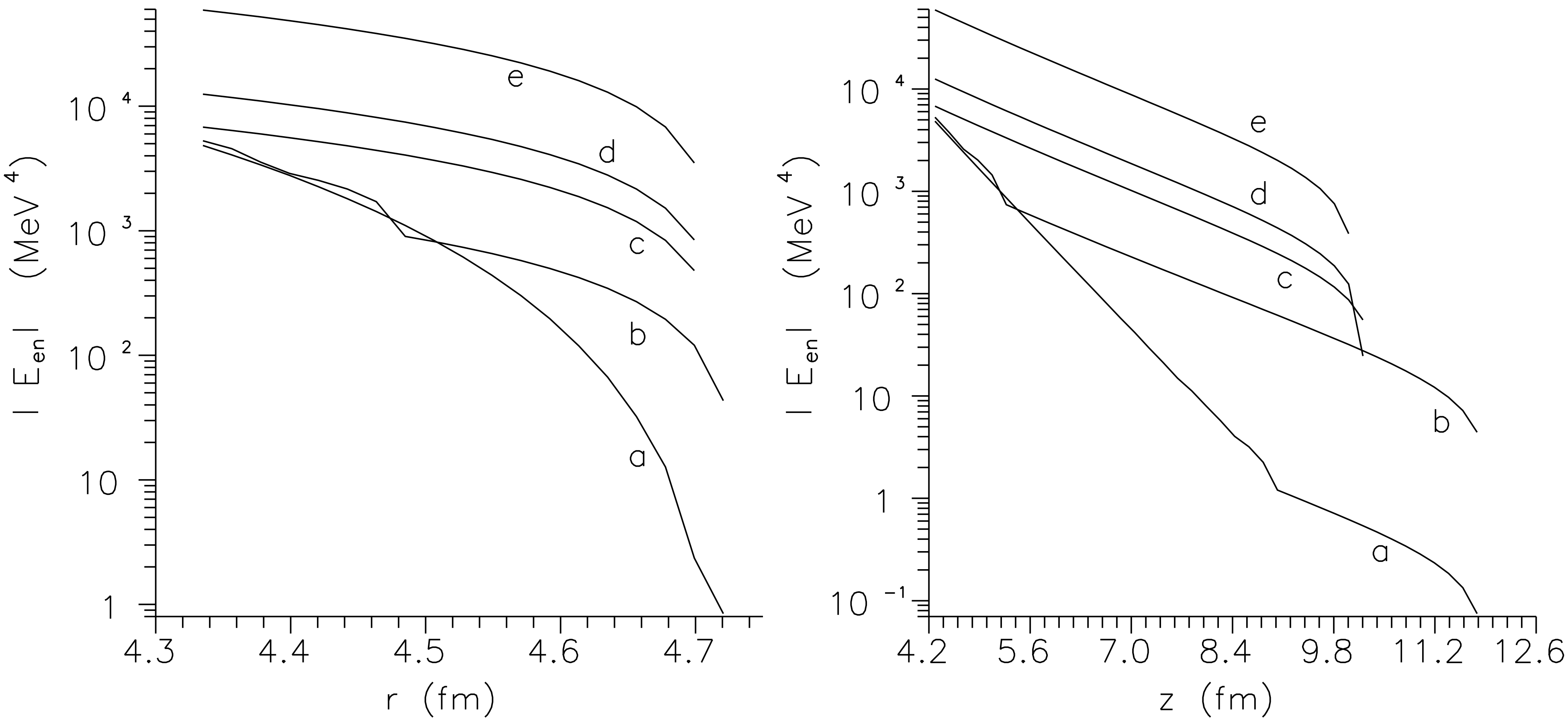
$$E_k = \frac{1}{V}\int \epsilon_k d^3r = \frac{2}{V}\frac{eB}{2\pi^2}\int d^3r \sum_{\nu=0}^{\nu_{max}} (2-\delta_{\nu 0}) \times \int_0^{p_F(r,z)} dp_z \left[(p_z^2 + m_\nu^2)^{1/2} - m_e\right] \quad (14)$$

ⓑ Cell-averaged electron nucleus interaction energy :

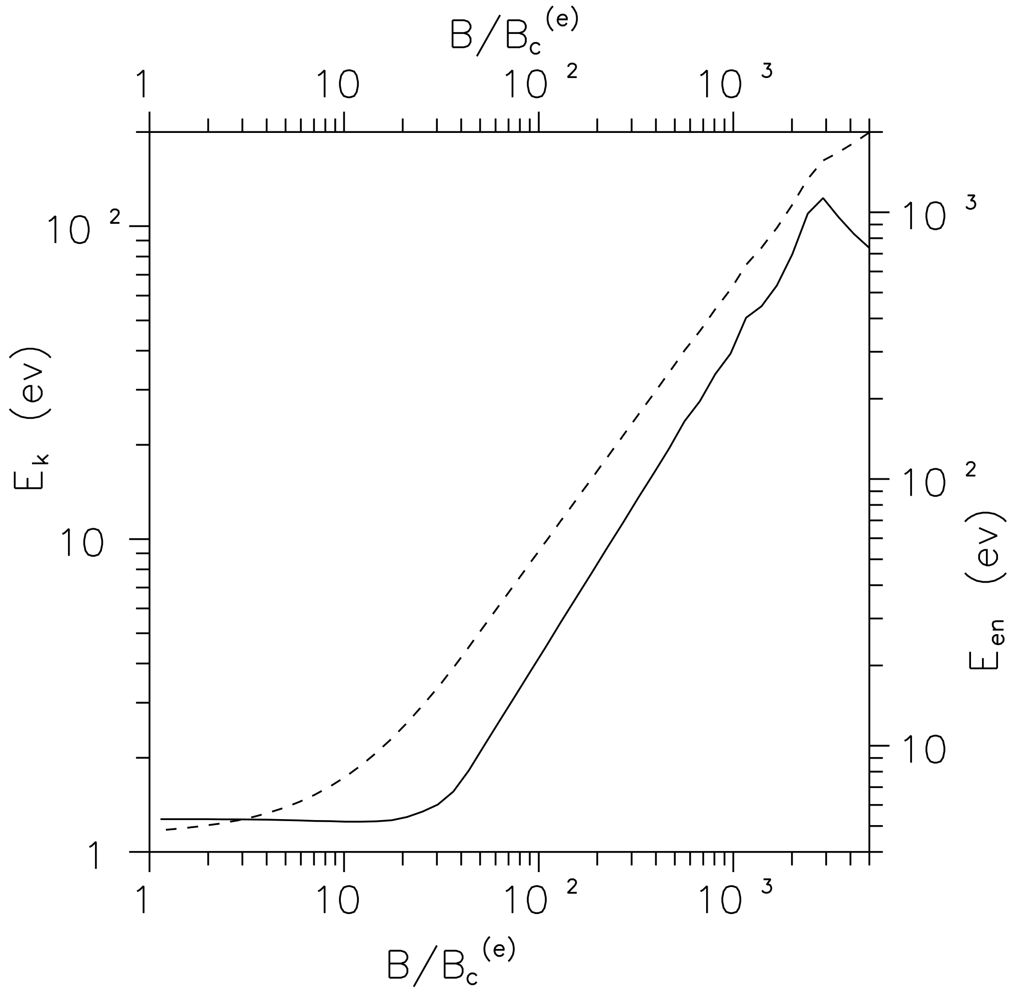
$$E_{en} = -\frac{2}{V} Z e^2 \int d^3r \frac{n_e(r,z)}{(r^2+z^2)^{1/2}} \quad (15)$$




Kinetic energy




$e-n$ Interaction energy




Variation of $E_{e-n}$ and $E_k$ with $B$

ⓒ Electron-electron direct interaction :

$$E_{ee}^{dir} = \frac{1}{V} e^2 \int d^3 r n_e(r,z) \int d^3 r' n_e(r',z') \times \frac{1}{\left[(r-r')^2 + (z-z')^2\right]^{1/2}} \tag{16}$$

ⓓ Electron-electron exchange interaction :

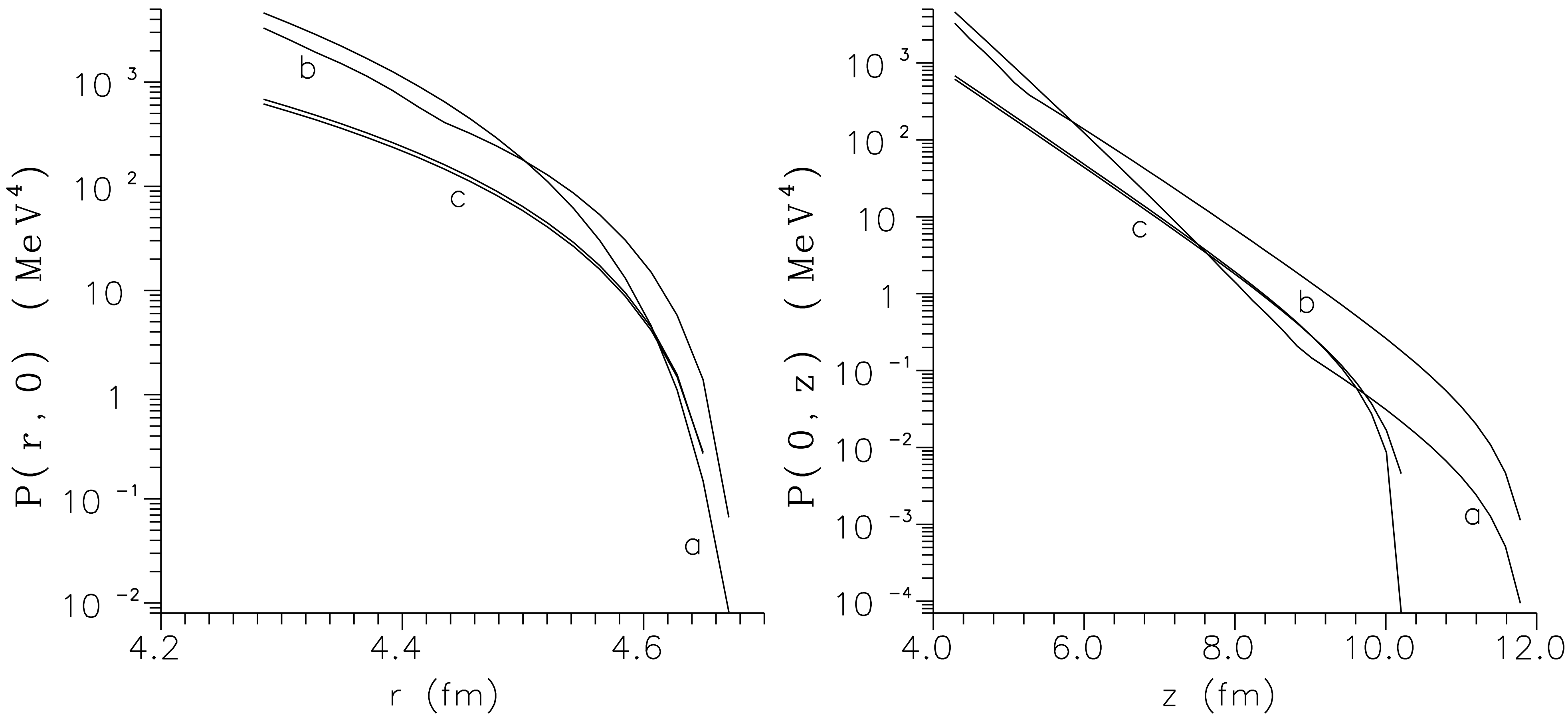
$$E_{ee}^{(ex)} = -\frac{e^2}{2} \sum_j \int d^3 r d^3 r' \frac{1}{\left[(r-r')^2 + (z-z')^2\right]^{1/2}} \times \bar{\psi}_i(r,z) \bar{\psi}_j(r',z') \psi_j(r,z) \psi_i(r',z') \tag{17}$$

ⓔ Kinetic pressure within the cell:

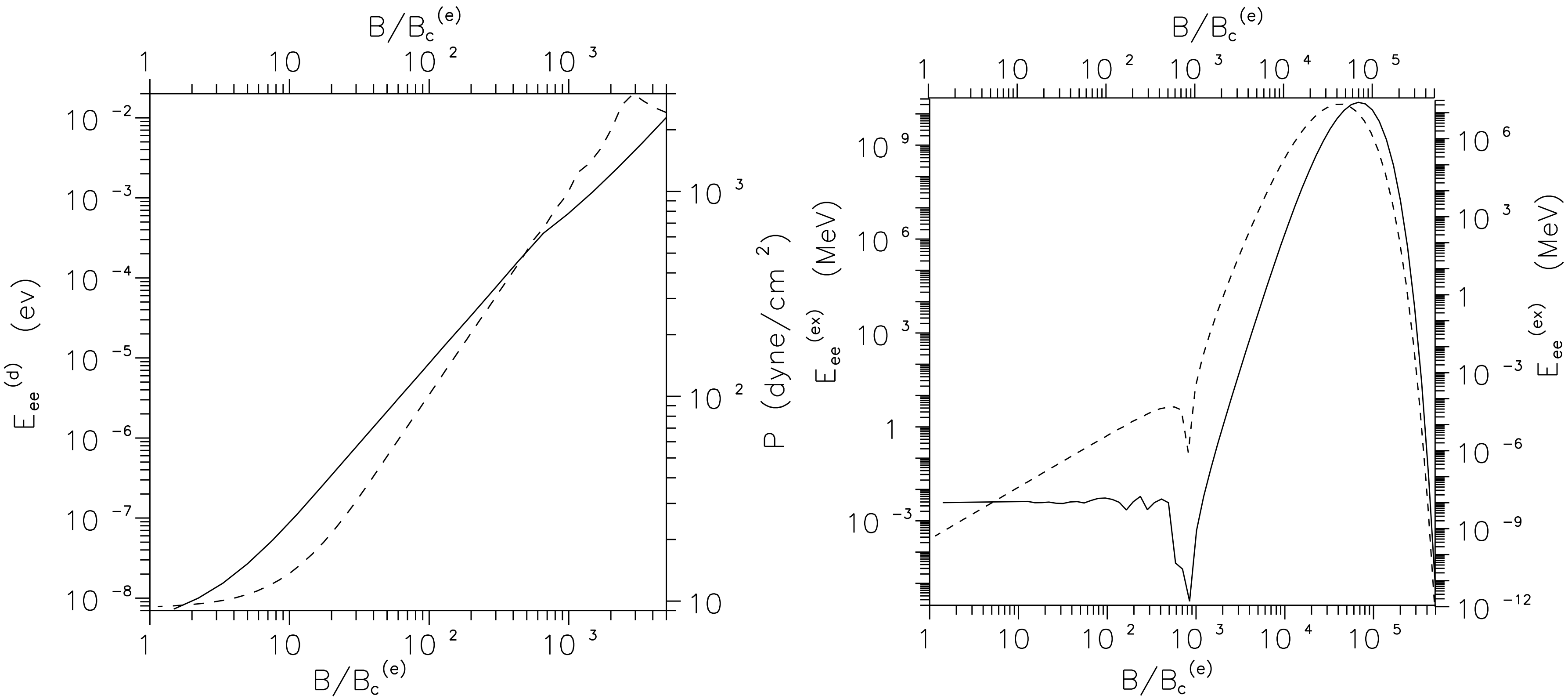
$$P(r,z) = \frac{eB}{4\pi^2} \sum_{\nu=0}^{\nu_{max}(r,z)} \left[ p_F (p_F^2 + m_\nu^2)^{1/2} - m_\nu^2 \ln \left\{ \frac{p_F + (p_F^2 + m_\nu^2)^{1/2}}{m_\nu} \right\} \right] \tag{18}$$

ⓕ Cell averaged $P$:

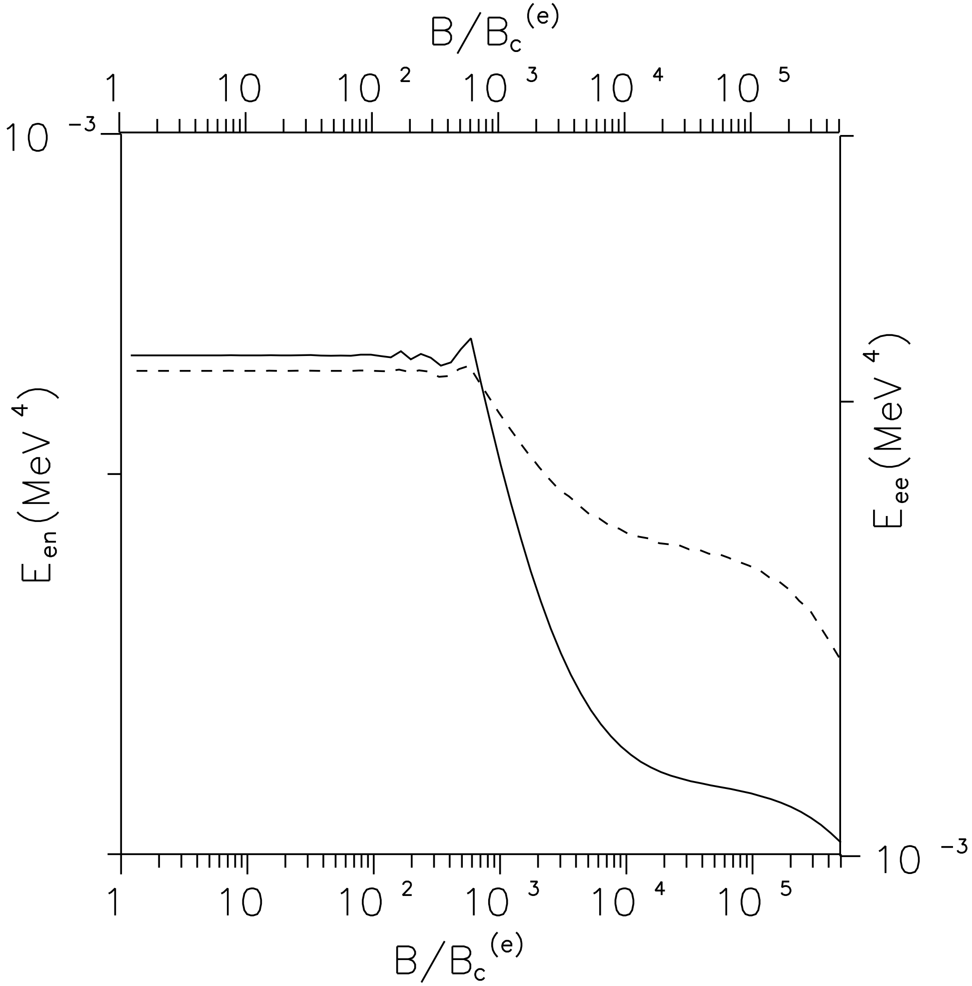
$$P = \frac{1}{V} \int P(r,z) d^3 r \tag{19}$$



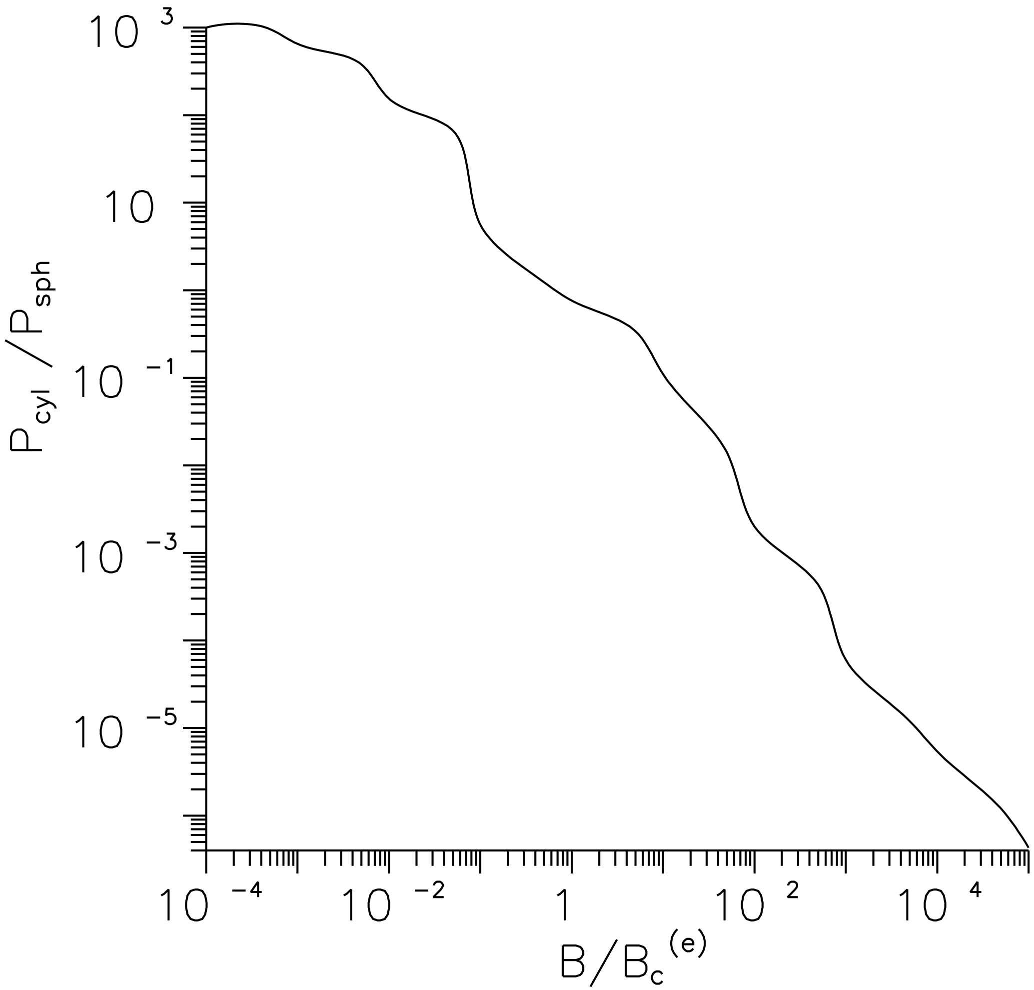

Variation of kinetic pressure within WS-cell




Variation direct energy, exchange energy ($e-e$ field type and Coulomb type) and cell averaged pressure with magnetic field




Variation $e-e$ and $e-n$ Coulomb interaction with magnetic field




Ratio of electron kinetic pressures in cylindrically deformed and spehrically symmetric cases.

## Conclusions:

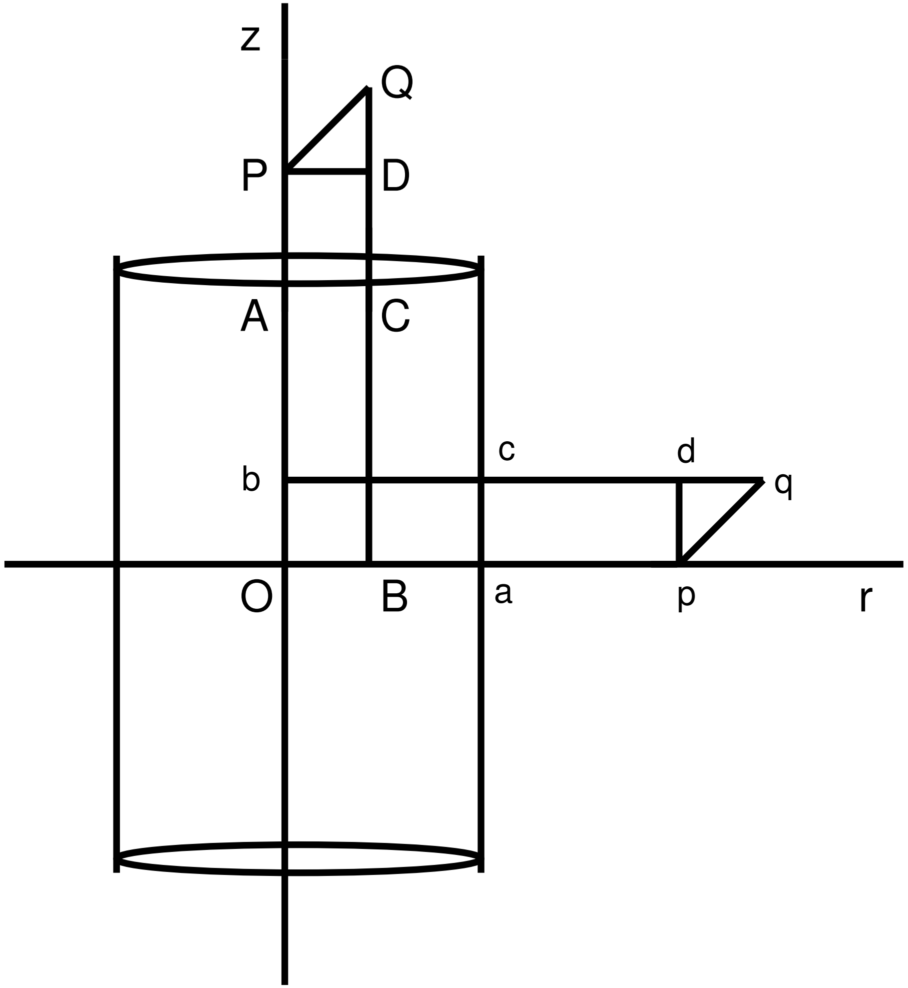
- Assumed cylindrical shape WS cells contract in presence of strong magnetic field.
- The Landau QN changes with both $r$ and $z$ coordinates and also with $B$.
- The inner crust matter is observed to be more stable if the electron distribution is cylindrical in nature.

# Variation of Work Function for Neutron Star Crustal Matter with Magnetic field

- To investigate the formation of magneto-sphere for a strongly magnetized neutron star or a magnetar, we need the variation of work function with strong magnetic field.

- Since the electrons are emitted mainly by cold or field emission process from the polar region by the action of strong electric field, created by rotating magnetic field, the work function plays a vital role in electron field emission from neutron star polar region.

- Since the emission occurs in strong magnetic field environment, we need to know the variation of work function with magnetic field.




Schematic diagram for the electron emission near the polar region of a neutron star.
Here the magnetic field is along $z$-axis.

- We have investigated electron emission along $z$-direction and also perpendicular direction to the magnetic field.

- Here the cylindrical region indicates the deformed electron distribution in presence of strong magnetic field.

- We have followed the mechanism of cold cathode emission.

- We assume that the frequency of incident electromagnetic wave is greater than the threshold frequency.

# A Longitudinal Emission

- Here the electrons are emitted along +z-direction.

- The electrostatic potential at Q produced by the emitted electron:

$$\phi^{(p)}(r,z) = e\int_{\xi=0}^{\infty} J_0(\xi r) \qquad \exp[\mp(z-h)\xi]d\xi, \quad \text{for } z > \text{ or } < h \tag{20}$$

Where $AP = h$, $PD = r$ and $CQ = z$.

- For charge distribution within the distorted WS cell (cylindrical region), we start with the cylindrical form of Poisson's equation:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial^2 z} = \lambda^2\psi \tag{21}$$

- The solution:

$$\psi(r,z) = \int_{\xi=0}^{\infty} J_0(\xi r)a(\xi) \quad \text{exp}[(\xi^2+\lambda^2)^{1/2}z]d\xi \quad \text{with } z < 0 \tag{22}$$

$a(\xi)$ is some unknown spectral function.

- The secondary field in coherence with $\phi^{(p)}(r,z)$ and $\psi(r,z)$:

$$\phi^{(s)}(r,z) = e\int_{\xi=0}^{\infty} J_0(\xi r)f(\xi) \quad \text{exp}(-\xi z)d\xi \quad \text{with } z > 0 \tag{23}$$

where $f(\xi)$ is again some unknown spectral function.

- This fictitious field pulls out electrons.

- Using the continuity conditions for tangential and transverse components of electric field and the displacement vector respectively:

$$E^{(t)} = E^{(p)t} + E^{(s)t} \text{ and } D^{\perp} = D^{(p)\perp} + D^{(s)\perp} \tag{24}$$

- We get

$$f(\xi) = \left[ \frac{1 - \left(1 + \frac{\lambda^2}{\xi^2}\right)^{1/2}}{1 + \left(1 + \frac{\lambda^2}{\xi^2}\right)^{1/2}} \right] \exp(-\xi h) \tag{25}$$

- Then we have the secondary potential

$$\phi^{(s)}(r, z) = e \int_{\xi=0}^{\infty} J_0(\xi r) \left[ \frac{1 - \left(1 + \frac{\lambda^2}{\xi^2}\right)^{1/2}}{1 + \left(1 + \frac{\lambda^2}{\xi^2}\right)^{1/2}} \right] \exp[-\xi(h + z)] d\xi \text{ for } z > 0 \tag{26}$$

- Hence the secondary field along the axial direction at $r = 0$:

$$
\begin{aligned}
E_z^{(s)} &= -\frac{\partial \phi^{(s)}}{\partial z} && (27) \\
&= e\int_{\xi=0}^{\infty}\left[\frac{1-\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}}{1+\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}}\right]\exp[-\xi(h+z)]\xi d\xi && (28)
\end{aligned}
$$

- The force acting on an electron at $z = h$ :

$$
F_z^{(s)} = eE_z^{(s)} = e^2\int_{\xi=0}^{\infty}\left[\frac{1-\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}}{1+\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}}\right]\exp(-2h\xi)\xi d\xi \qquad (29)
$$

- The work done in pulling out an electron from just inside the metal surface to infinity:

$$W_f = -\int_0^{\infty} F_z^{(s)} dh \tag{30}$$

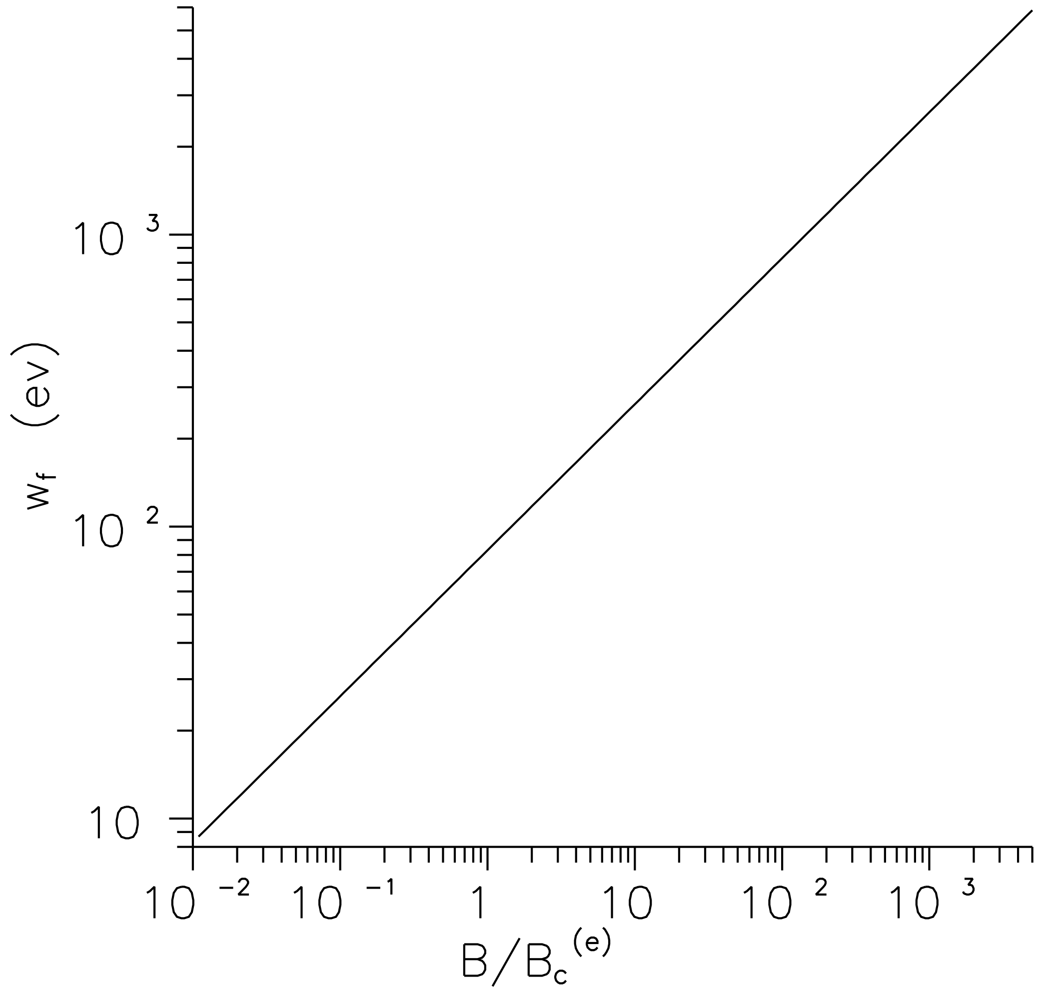
$$= -\frac{e^2}{2}\int_{\xi=0}^{\infty} \frac{1-\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}}{1+\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}} d\xi \tag{31}$$

- After evaluation we get :

$$W_f = \frac{\lambda}{3}e^2 = \frac{1}{3}\left(\frac{2B}{\pi B_c^{(e)}}\right)^{1/2} m_e e^3 \quad (32)$$

- Where $B_c^{(e)} \approx 4.43 \times 10^{13}$G, the typical field strength to start populating the Landau levels for electrons in the relativistic scenario.

- This equation gives the variation of work function with the strength of magnetic field $(\sim B^{1/2})$ associated with the emission of electrons along the direction of the magnetic field.




The variation of longitudinal part of work function with magnetic field.

- From our investigation it is obvious that the work function increases with the increase in magnetic field strength.

- Physically it means that with the increase in magnetic field strength, electrons become more strongly bound inside the crustal matter.

- As a consequence the electron emission current becomes extremely small in the case of neutron stars with ultra-strong magnetic field.

- $\Longrightarrow$ charge density in the magneto-sphere will be low enough $\Longrightarrow$ intensity of radio wave emission will be extremely low.

# B Transverse Emission

- We now consider the emission of electrons in the direction transverse to the magnetic field direction.
- Here $p$ is the position of an emitted electron.
- The electronic potential at $q$:

$$\phi^{(p)}(r,z) = e\int_{\xi=0}^{\infty} J_0[\xi(r-r_0)]\exp(\mp z\xi)d\xi \quad \text{for } z>0 \text{ or } z<0 \tag{33}$$

Where $ap = r_0$, $cq = r$ and $pd = z$.

- Whereas the form of $\psi(r,z)$ remains unchanged.

- Using some of the properties of $J_0$ we have

$$\psi(r,z) = \sum_{k=-\infty}^{+\infty} J_k(\xi r_0)\int_{\xi=0}^{\infty} J_k[\xi(r+r_0)]a(\xi) \exp[-(\xi^2+\lambda^2)^{1/2} \mid z \mid] d\xi \tag{34}$$

- Similarly we have

$$\phi^{(p)}(r,z) = e \sum_{k=-\infty}^{+\infty} J_k(2\xi r_0) \int_{\xi=0}^{\infty} J_k[\xi(r+r_0)] \exp(-\xi \mid z \mid) d\xi \qquad (35)$$

and

$$\phi^{(s)}(r,z) = e \sum_{k=-\infty}^{+\infty} J_k(\xi r_0) \int_{\xi=0}^{\infty} J_k[\xi(r+r_0)] f(\xi) \exp(-\xi \mid z \mid) d\xi \qquad (36)$$

- Then from the continuity conditions on the curved surface along $r$ direction, putting $z = 0$, $r = R$ and redefining the spectral function $a(\xi) = a(\xi)/e$:

$$\int_{\xi=0}^{\infty} \xi d\xi \sum_{k=-\infty}^{+\infty} J_k'[(R+r_0)\xi]\{a(\xi) J_R(\xi r_0) - J_k(2\xi r_0) - J_k(\xi r_0) f(\xi)\} = 0 \qquad (37)$$

- Similarly the continuity condition along $z$-direction on the curved surface at $z = 0$ and $r = R$:

$$\begin{aligned} &\int_{\xi=0}^{\infty} \sum_{k=-\infty}^{+\infty} J_k(\xi r_0) J_k[(R + r_0)\xi] a(\xi)(\xi^2 + \lambda^2)^{1/2} d\xi \\ = &\int_{\xi=0}^{\infty} \sum_{k=-\infty}^{+\infty} J_k(2\xi r_0) J_k[\xi(R + r_0)]\xi d\xi \\ + &\int_{\xi=0}^{\infty} \sum_{k=-\infty}^{+\infty} J_k(\xi r_0) J_k[\xi(R + r_0)] f(\xi)\xi d\xi \end{aligned} \tag{38}$$

- Hence after some simple algebraic manipulation we have:

$$\phi^{(s)}(r, z) = e \int_{\xi=0}^{\infty} J_0(\xi r) \left[ \frac{F(1 + \frac{\lambda^2}{\xi^2})^{1/2} - G}{1 - \left(1 + \frac{\lambda^2}{\xi^2}\right)^{1/2}} \right] \exp(-\xi z) d\xi \tag{39}$$

Where

$$F = \frac{J_1[(R - r_0)\xi]}{J_1(R\xi)} \text{ and } G = \frac{J_0[(R - r_0)\xi]}{J_0(R\xi)} \tag{40}$$

- we have used some well-known relations of $J_k$ and $J'_k$.

- Hence the corresponding electro-static field at $(R, z = 0)$:

$$E^{(s)}(R,0) = -\frac{\partial \phi^{(s)}(r,z)}{\partial r}$$

$$= \; e\int_{\xi=0}^{\infty} \frac{J_1[(R-r_0)\xi]\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2} - \frac{J_0[(R-r_0)\xi]J_1(R\xi)}{J_0(R\xi)}}{1-\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}}\xi d\xi \quad (41)$$

- The work function associated with the emission:

$$W_f = -e\int_{R}^{\infty} E^{(s)}(R,0)dr_0 \quad (42)$$

- Now using some well-known integrals for $J_0$ and $J_1$ we finally get

$$W_f = -e^2 \int_{\xi=0}^{\infty} \frac{\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2} - \left[\frac{J_1(R\xi)}{J_0(R\xi)}\right]}{1-\left(1+\frac{\lambda^2}{\xi^2}\right)^{1/2}} d\xi \tag{43}$$

- This integral for work function has been obtained numerically and found to be diverging in nature.

# Conclusions

- Work function is showing anisotropic nature in presence of strong magnetic field.
- Along the field direction, work function $W_f \propto B^{1/2}$, $\Longrightarrow$ the electron will be more strongly bound inside the crustal region if the magnetic field is strong enough.
- In the transverse direction the work function is found to be large enough compared to the longitudinal part.
- In the extreme case, transverse part of the work function diverge for ultra strong magnetic field strength.
- We expect that such anisotropic behavior and identical type variation of work function in the strong magnetic field environment should also be there in the case of laboratory samples.

# Fowler-Nordheim (F-N) Electron Cold Emission Formalism in Presence of Strong Magnetic Field

- Unlike other electron emission process, in cold emission, electrons are pulled out from metal surface by strong electric field.

- This is a purely quantum mechanical tunneling process through surface barrier.

- Formalisms for both non-relativistic and relativistic versions of field emission of electrons in presence of a strong quantizing magnetic field, relevant for strongly magnetized neutron stars or magnetars, are developed.

- In the non-relativistic scenario, where electrons obey the Schrödinger equation, we have noticed that when Landau levels are populated for electrons in presence of strong quantizing magnetic field the transmission probability exactly vanishes unless the electrons are spin polarized.

- On the other hand, the cold electron emission under the influence of strong electrostatic field at the poles is totally forbidden from the surface of those compact objects for which
  (i) electrons are unpolarized or spin is neglected and
  (ii) the surface magnetic field strength is $\gg 10^{15}$G.

- Whereas in the relativistic case, where the electrons obey Dirac equation, the presence of strong quantizing magnetic field completely forbids the emission of electrons from the surface of compact objects if $B > 10^{13}$G, i.e., when the Landau levels are populated.

# A Non-relativistic scenario

- In presence of strong magnetic field the electron distribution at the crustal region gets deform.

- Instead of spherical, the distribution is assumed to be ellipsoidal in nature.

- In this investigation we have assumed, for the sake of simplicity, cylindrical type distribution of electrons around positively charged nuclei.

- We have Followed Fowler-Nordheim's epoch making paper in the proceeding of the Royal Society of London, to investigate the electron emission from the poles by cold emission mechanism.

- The strength of electric field at the poles of a strongly magnetized neutron star is $F \sim 2 \times 10^8 p^{-1} B_{12} V cm^{-1}$, $B_{12} \longrightarrow$ magnetic field strength in units of $10^{12}$ at the poles and $P \longrightarrow$ time period in second.

- Strong magnetic field is also indirectly responsible for magnetosphere formation:

- High energy emitted electrons produce curvature $\gamma$-photons.

- This photons can create $e^- - e^+$-pairs if energy $> 2 \times m_e$.

- To solve the problem analytically, following F-N we take triangular shape surface potential $V(z) = C - Fz$, $C \longrightarrow$ is the constant surface barrier and $F \longrightarrow$ driving force for electron emission.

- The electrons before emission satisfy the Schrödinger equation in cylindrical co-ordinate (We assume $(\hbar = c = 1)$):

$$-\frac{1}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}\right] - \frac{ieB}{2m}\frac{\partial\psi}{\partial\theta} + \left(\frac{e^2B^2\rho^2}{8m_e} - E\right)\psi = 0 \quad (44)$$

- Electrons just liberated, satisfy the equation:

$$-\frac{1}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}\right]-\frac{ieB}{2m}\frac{\partial\psi}{\partial\theta}+\left(\frac{e^2B^2\rho^2}{8m_e}-E+V(z)\right)\psi=0 \tag{44a}$$

- With separable solution:

$$\psi(\rho,\theta,z)=\phi_{n_\rho,m}(\rho,\theta)f_\nu(z) \tag{45}$$

- The radial part along with $\theta$ dependency is given by

$$\begin{aligned}\phi_{n_\rho,m}(\rho,\theta) &= \frac{\exp(im\theta)}{(2\pi)^{1/2}}\rho_0^{-1-|m|}\left[\frac{(\mid m\mid+n_\rho)!}{2^{|m|}n_\rho!\mid m\mid!}\right] \\ &\times\ \rho^{|m|}\exp\left(-\frac{\rho^2}{4\rho_0^2}\right)L_{n_\rho,m}\left(\frac{\rho^2}{2\rho_0^2}\right)\end{aligned} \tag{46}$$

$\rho_0 \longrightarrow$ is the Larmor radius.

- Energy without spin:

$$E = \frac{p_z^2}{2m_e} + \mu_B B(2n_\rho + \nu \pm \nu + 1) \tag{47}$$

- Energy with spin:

$$E = \frac{p_z^2}{2m_e} + \mu_B B(2n_\rho + \nu \pm \nu + 1) \mp \mu_B B \tag{48}$$

$\mu_B \longrightarrow$ Bohr magneton.

- Electron just emitted, equation along $z$-axis:

$$-\frac{1}{2m}\frac{d^2 f_0}{dz^2} + V(z) f_0 = (E - \mu_B B)\, f_0 = w f_0 \tag{49}$$

- Electrons within the crustal matter satisfy:

$$\frac{d^2 f_0}{dz^2} + w_k^2 f_0 = 0 \tag{50}$$

- The solution of this equation:

$$f_0 = \frac{1}{w_k^{1/2}} \left[ a \exp(i w_k z) + a' \exp(-i w_k z) \right] \tag{51}$$

- For electrons just tunneled out, the solution is

$$f_0(y) = y^{1/2} H_{1/3}^{(2)} \left( \frac{2}{3} y^{3/2} \right) \tag{52}$$

- We define the quantity:

$$Q = \frac{2}{3} (2 m_e F)^{1/2} \left( \frac{C - w}{F} \right)^{3/2} \tag{52a}$$

- The transmission probability for electrons:

$$D(w) = \frac{|\, a \,|^2 - |\, a' \,|^2}{|\, a \,|^2} \tag{53}$$

- The field emission current for electrons:

$$R = \frac{eB}{2\pi^2}\int_0^\infty f(w)D(w)\frac{p_z}{m_e}dp_z \qquad (53a)$$

- From the continuity of wave function and their derivatives at the interface:

$$\begin{aligned} D(w) &\approx \frac{[w(C-w)]^{1/2}}{C}\exp\left(-\frac{4}{3}(2m_eF)^{1/2}\left(\frac{C-w}{F}\right)^{3/2}\right) \\ &= \frac{[w(C-w)^{1/2}]}{C}\exp\left(-2Q\right) \end{aligned} \qquad (54)$$

- Obviously magnitude of $D$ strongly depends on the magnitude of $Q$.

- This factor $\approx \infty$ in absence of electron spin $\longrightarrow$ makes transmission probability for electrons $\rightarrow 0$.

- To study the effect of magnetic field on cold electron current from the poles, we assume following F-N:
  $C = \mu_e + w_f$, $\mu_e \longrightarrow$ electron chemical potential and
  $w_f = w_c \times (B/B_c^{(e)})^{1/2}$eV, $w_c \approx 82.93$.

- Then one can express $Q$ in the following form:

$$Q \approx h_f^{1/2}(\beta \times 0.5 + w_c h_f^{-1/2} \times 10^{-6})^{3/2} \times 10^7 = Q_B + Q_{wf} \tag{55}$$

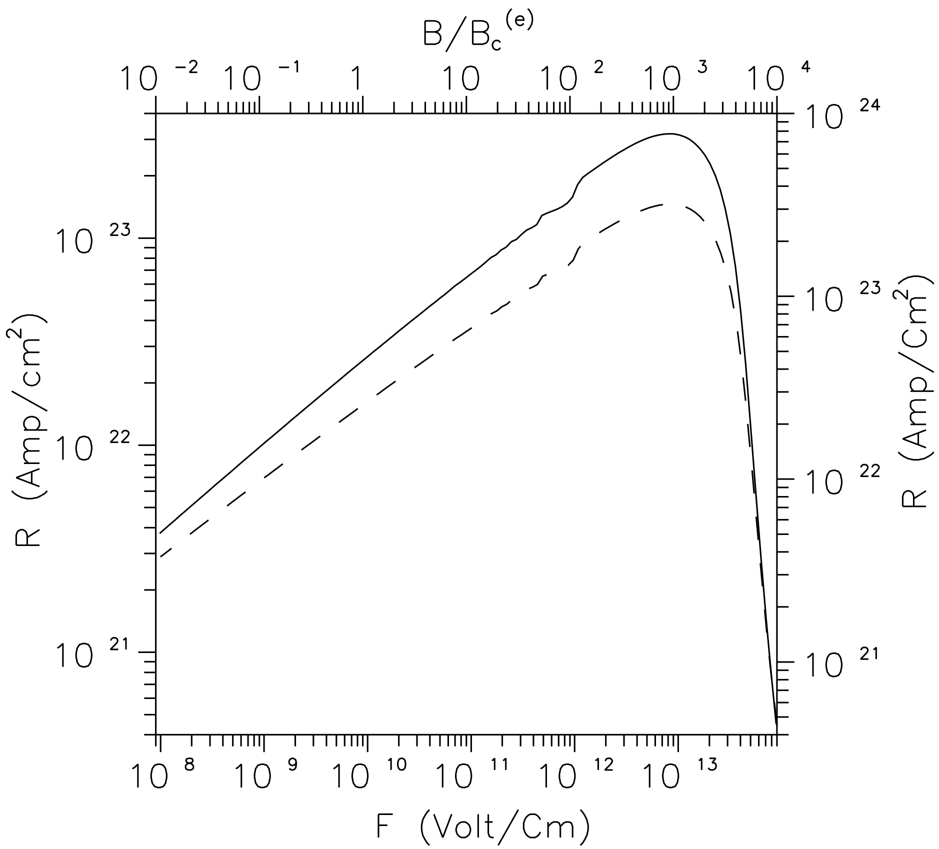
- $\beta = 0 \longrightarrow$ conventional electron spin polarization, $\beta = 1 \longrightarrow$ with no electron spin and $\beta = 2 \longrightarrow$ opposite spin polarization.

- For $\beta = 1$ or $2 \longrightarrow Q_B$ extremely large $\Longrightarrow$ $Q$ also large enough ($\longrightarrow \infty$) $\Longrightarrow$ transmission current $R$ vanishingly small.

- For $\beta = 0$, $Q_B = 0 \Longrightarrow Q = Q_{wf}$ is finite $\Longrightarrow$ transmission current is finite.

- In this case the variation of transmission current with electric field (also with magnetic field):

$$R = 0.26F_{24}^{1/2}\exp(-9.8\times F_{14}) \tag{56}$$




The solid curve is for the variation of electron field current with electric field and the dashed one is for the variation of that with magnetic field.

## B Relativistic scenario

- The potential is introduced in field theoretic manner.

- The radial part of Dirac equation satisfied by the upper component:

$$\left[\beta_\lambda + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - k^2\rho^2\right] R(\rho) = 0 \tag{57}$$

$\beta_\lambda^2 = E^2 - m^{*2} - p_z^2 + 2\lambda k$ with $\lambda = \pm 1 \Longrightarrow$ up and down spin states.
$k = eB/2$ and $m^* = m + C - Fz$ with $m^* = m$ for electrons when $V(z) = 0$.

- The solution of the above equation:

$$R(\rho) = N \exp\left(-\frac{t}{2}\right) L_n(t) \tag{58}$$

- $L_n(t) \longrightarrow$ Legendre Polynomial, $t = k^2\rho^2$ and
energy eigen value $E_\nu = (P_z^2 + m^2 + 2\nu eB)^{1/2}$.

- Motion along $z$-axis:

$$\frac{d^2 f_\nu}{dz^2} + (E^2 - m^{*2} - 2\nu eB) f_\nu = 0 \tag{59}$$

- Using the transformation:

$$z = \frac{m + C - uF^{1/2}}{F} \tag{60}$$

- We have

$$\frac{d^2 f_\nu}{du^2} + (\alpha^2 - u^2) f_\nu = 0 \longrightarrow 1 - \mathrm{D\ QM\ HO} \tag{61}$$

  Where $\alpha = (E^2 - 2\nu eB)^{1/2}/F^{1/2}$ with $\alpha^2 = 2l$
  we have $E^2 = 2(\nu eB + lF)$ and

$$f_{\nu,l} = \tilde{N} \exp\left(-\frac{u^2}{2}\right) H_l(u) = (f_\nu)_I \quad \text{(say)} \tag{62}$$

- Electrons within the crustal matter ($V(z) = 0$):

$$\frac{d^2 f_\nu}{dz^2} + \alpha^2 f_\nu = 0 \tag{63}$$

Here $\alpha = (E^2 - m^2 - 2\nu eB)^{1/2}$

- Solution:

$$f_\nu = \frac{1}{\alpha^{1/2}} \left[ a \exp(i\alpha z) + a' \exp(-i\alpha z) \right] = (f_\nu)_{II} \quad \text{(say)} \tag{64}$$

- Continuity at $z = 0$:

$$(f_\nu(0))_{II} = (f_\nu(0))_I \quad \text{and} \quad \left(f_\nu'(0)\right)_{II} = \left(f_\nu'(0)\right)_I \tag{65}$$

- Hence:

$$a + a' = N\alpha^{1/2} \exp\left(-\frac{u_0^2}{2}\right) H_l(u_0) \tag{66}$$

- and

$$i\alpha^{1/2}(a - a') = N \exp\left(-\frac{u_0^2}{2}\right)\left[u_0 H_l(u_0) - 2l H_{l-1}(u_0)\right] \quad (67)$$

Where $u_0 = u(z = 0) = m^*/F^{1/2}$

- Defining

$$a + a' = X \text{ and } a - a' = iY, \longrightarrow X, Y \text{ Real Quantities} \quad (68)$$

- Hence the transmission coefficient

$$T = \frac{|a|^2 - |a'|^2}{|a|^2} = 0 \quad (69)$$

## Conclusions :

- Electron emission is allowed if electron spin is considered.
- To have non-zero transmission current, in addition, electrons must have conventional spin polarization.
- For extremely high magnetic field, $w_f \longrightarrow \infty$, transmission current $\longrightarrow 0$.
- In the strong magnetic field case, charge density in the magneto-sphere will be extremely low.
- For the relativistic scenario electrons can not be extracted whatever be the strength of surface electric field at the poles if $B > 10^{13}$G, i.e., the Landau levels for the electrons are populated.
- Our theoretical results do not contradict the known observational data on electron emission from magnetars.

# Relativistic Field Emission of Electrons in $1+1$-Dimension

- In this part we have presented a formalism for electron emission in $1+1$-dimension with relativistic scenario.

- We assume $1$-Dimensional Dirac equation satisfied by electrons at the polar region of a strongly magnetized neutron star.

- We have included a triangular type potential barrier at the surface near polar region.

- Further, to avoid Klein paradox, the potential is introduced in a field theoretic manner.

# Relativistic Emission Model in Presence of Scalar Potential Barrier

- We use the following form of Dirac equation in presence of scalar potential $V(x)$:

$$[\vec{\alpha}.\vec{p} + \beta(m + V(x))]\,\psi = E\psi \tag{70}$$

- Where as before $V(x) = C - eFx \longrightarrow C$ is the surface barrier and $F$ (absorbing the electron charge, we express $eF$ by $F$) is the constant electric field $\longrightarrow$ acts as the driving force for electron emission.

- We use the following representation:

$$\alpha = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \beta = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{71}$$

- We define: $m^* = m + C \longrightarrow$ effective electron mass.

- Spinor solution:

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \tag{72}$$

- Hence

$$-\frac{dv}{dx} + (m^* - Fx)v = Eu \tag{73}$$

$$\frac{du}{dx} + (m^* - Fx)u = Ev \tag{74}$$

- After eliminating $v$ and with the transformation:

$$\xi = F^{1/2}\left(\frac{m^*}{F} - x\right) \tag{75}$$

- We have

$$-\frac{d^2u}{d\xi^2} + \xi^2 u = \left(\frac{E^2}{F} - 1\right) u \tag{76}$$

- This equation may be re-arranged in the following form:

$$\frac{d^2u}{d\xi^2} + (\lambda - \xi^2)u = 0 \tag{77}$$

$\Longrightarrow$ equation for $1$-D HO with $\lambda = 1 + 2\nu$,

- we have

$$E^2 = 2(\nu + 1)F$$

and

$$u = N\frac{E}{F^{1/2}} \exp\left(-\frac{\xi^2}{2}\right) H_\nu(\xi) = u_I \text{ (say)} \tag{78}$$

- Within the crustal matter, Dirac equation in $1+1$-dimension:

$$(\vec{\alpha}.\vec{p}+\beta m)\,\psi = E\psi \tag{79}$$

- This equation can be transformed as

$$\frac{d^2u}{dx^2}+w_k^2u=0 \tag{80}$$

where $w_k=(E^2-m^2)^{1/2}$ is the free electron momentum.

- Following Fowler-Nordheim:

$$u=\frac{1}{w_k^{1/2}}\left[a\exp(iw_kx)+a'\exp(-iw_kx)\right]=u_{II}\ \ (\text{say}) \tag{81}$$

- Assuming the interface at $x=0$:

$$u_{II}(0)=u_I(0)\quad\text{and}\quad u'_{II}(0)=u'_I(0) \tag{82}$$

- Hence

$$a + a' = N w_k^{1/2} \frac{E}{F^{1/2}} \exp\left(-\frac{\xi_0^2}{2}\right) H_\nu(\xi_0) \tag{83}$$

and

$$i w_k^{1/2}(a - a') = NE \exp\left(-\frac{\xi_0^2}{2}\right) [\xi_0 H_\nu(\xi_0) - 2\nu H_{\nu-1}(\xi_0)] \tag{84}$$

- These two equations may be arranged in the following form:

$$a + a' = X \text{ and } a - a' = iY \tag{85}$$

where $X$ and $Y$ are two real quantities.

- Hence

$$T = 1 - \frac{|a'|^2}{|a|^2}, \tag{86}$$

vanishes exactly.

## Conclusion:

- This investigation shows that in $1+1$-dimension the transmission coefficient of field electron emission vanishes exactly in the relativistic scenario, in presence of scalar type surface potential at the poles.

- We have also notice that transmission coefficient is also zero for vector type surface potential, of course it has no relevance in astrophysics