Schrödinger Equation in Uniformly Accelerated Frame

for

Departmental Seminar

- The Work is Based on the Principle of Equivalence. An accelerated frame of reference is equivalent to a frame at rest in presence of gavitational field of strength equal to the acceleration of the frame.

- The gravitational field is supposed to be produced by a strong gravitating object-say a black hole.

- A black hole is defined as a region of space-time that can not communicate with the external universe.

- The Event Horizon of a black hole is the boundary region or the surface of the black hole.

Rindler Coordinate

- The Rindler space-time coordinates are the just an uniformly accelerated frame transformation of the Minkowski metric of special relativity.

$$ct = \left(\frac{c^2}{g} + x'\right) \sinh\left(\frac{gt'}{c}\right) \quad \text{and} \quad x = \left(\frac{c^2}{g} + x'\right) \cosh\left(\frac{gt'}{c}\right)$$

- The inverse relations:

$$ct' = \frac{c^2}{2g} \ln\left(\frac{x + ct}{x - ct}\right) \quad \text{and} \quad x' = (x^2 - (ct)^2)^{1/2} - \frac{c^2}{g}$$

- Hence:

$$x \pm ct = \left(\frac{c^2}{\alpha} + x'\right) \exp(\pm \alpha t'/c)$$

- The motion is assumed to be rectilinear and along x-direction, $dy' = dy$ and $dz' = dz$.

- Then we have:

$$ds^2 = (dct)^2 - (dx)^2 = \left(1 + \frac{\alpha x'}{c^2}\right)^2 (dct')^2 - (dx')^2$$

- In 3-D Rindler coordinate transformation the Minkowski line element becomes

$ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2$ to $ds^2 = \left(1 + \frac{gx'}{c^2}\right)^2 d(ct')^2 - dx'^2 - dy'^2 - dz'^2$

- The single particle classical Lagrangian:

- The action integral may be written as:

$$S = -\alpha_0 \int_a^b ds \equiv \int_a^b L dt$$

Hence for $\alpha_0 = m_0 c^2$

$$L = -m_0 c^2 \left[\left(1 + \frac{\alpha x}{c^2} \right)^2 - \frac{v^2}{c^2} \right]$$

- where $\alpha = g$ is the uniform acceleration of the particle along x-direction

- $v = u_x$, the particle velocity and m_0 is the rest mass of the particle.

- The three momentum vector of the particle can then be written as

$$\vec{p} = \frac{\partial L}{\partial \vec{v}}$$

$$\vec{p} = \frac{m_0 \vec{v}}{\left[\left(1 + \frac{\alpha x}{c^2}\right)^2 - \frac{v^2}{c^2}\right]^{1/2}}$$

- The Hamiltonian of the particle:

$$H = \vec{v}.\vec{p} - L$$

$$H = m_0 c^2 \left(1 + \frac{\alpha x}{c^2}\right) \left(1 + \frac{p^2}{m_0^2 c^2}\right)^{1/2}$$

- Quantum mechanical picture $\longrightarrow$ the classical dynamical variables x, $\vec{p}$ and H are treated as operators $\Longrightarrow$ the commutation relations

$$[x, p_x] = i\hbar \text{ and } [x, p_y] = [x, p_z] = 0$$

- The Schrödinger equation for the particle:

$$H\psi = \left(1 + \frac{\alpha x}{c^2}\right)\left(m_0 c^2 + \frac{p^2}{2m_0}\right)\psi = E\psi$$

Hence

$$\left(m_0 c^2 - \frac{\hbar^2}{2m_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)\psi = \frac{1}{1 + \frac{\alpha x}{c^2}} E\psi$$

- Since there is x-dependent term $\Longrightarrow$ the separable form of the solution:

$$\psi(x, y, z) = N X(x) \exp\left(-\frac{i p_y y}{\hbar}\right) \exp\left(-\frac{i p_z z}{\hbar}\right)$$

- On substituting $\psi(x, y, z)$:

$$\left(m_0 c^2 + \frac{p_y^2}{2m_0} + \frac{p_z^2}{2m_0}\right) X - \frac{\hbar^2}{2m_0}\frac{d^2 X}{dx^2} = \frac{1}{1 + \frac{\alpha x}{c^2}} E X$$

- The quantity within the first bracket on the left hand side is defined as $E_\perp$, the orthogonal part of the total energy, including the rest mass energy $\Longrightarrow$

$$E_\perp X - \frac{\hbar^2}{2m_0}\frac{d^2 X}{dx^2} = \frac{1}{1 + \frac{\alpha x}{c^2}} E X$$

• Introduce a new variable $u = 1 + \alpha x/c^2$ and define two constant quantities

$$a = \frac{2m_0 c^4}{\hbar^2 \alpha^2} E \quad \text{and} \quad b = \frac{8m_0 c^4}{\hbar^2 \alpha^2} E_{\perp},$$

• Above equation takes the following form

$$\frac{d^2 X}{du^2} + \frac{a}{u} X - \frac{b}{4} X = 0$$

• Defining $v = b^{1/2} u$ as a new variable and $\gamma = ab^{-1/2}$ as another constant quantity, $\Longrightarrow$

$$\frac{d^2 X}{dv^2} + \left(-\frac{1}{4} + \frac{\gamma}{v}\right) X = 0$$

• On comparing this differential equation with that satisfied by the Whittaker function $M_{k,\mu}(x)$ given by

$$\frac{d^2}{dx^2} M_{k,\mu}(x) + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right) M_{k,\mu}(x) = 0$$

• we have $X(v) \equiv M_{k,\mu}(v)$, for $x = v$, $\mu = 1/2$ and $k = \gamma$. Then one can write down the solution in the explicit form as

$$X(v) = M_{\gamma,\frac{1}{2}}(v) = \exp\left(-\frac{v}{2}\right) v M(1 - \gamma, 2, v)$$

- where $M(a,c,x) = {}_1F_1(a;c;x)$, the Confluent Hypergeometric function.

- From the knowledge of special functions, clearly, $M(1-\gamma, 2, v)$ will be a polynomial and becomes zero for $v \longrightarrow \infty$, if the parameter $1-\gamma$ is 0 or a negative integer, i.e., $\gamma = n$, for $n = 1, 2, 3......$, the positive integers.

- Under such restricted situation, the solution can also be expressed in terms of Associated Laguerre function.

- This alternative form of wave function is then given by

$$X(v) = \exp\left(-\frac{v}{2}\right) v L^{1}_{\gamma-1}(v)$$

- The parameter γ is again have to be non-zero positive integer.

- It is exactly identical with one-dimensional hydrogen atom.

•The quantized form of energy eigen value of the particle in an uniformly accelerated frame is given by

$$E_n = n\hbar\frac{\alpha}{c} = n\hbar\omega \ \text{(say)}$$

with

$$\omega = \frac{\alpha}{c}$$

Where we have neglected p_y and p_z for a purely one-dimensional condition.

•Then accordingly $E_\perp = m_0 c^2$, the rest mass energy of the particle.

•Surprisingly the differential equation satisfied by the particle is exactly identical with the equation for an one-dimensional hydrogen atom, whereas the quantized energy levels look like that of the energy levels for an one-dimensional quantum harmonic oscillator.

•The energy levels vary linearly with the quantum number n, instead of $1/n^2$, where the last one is the case for an one-dimensional hydrogen atom.

•The energy levels are observed to be independent of particle rest mass. It depends only on the acceleration α of the frame.

•It is obvious that $\omega \longrightarrow 0$ as $\alpha \longrightarrow 0$.

• Further, unlike the harmonic oscillator case, the minimum energy of the particle or the ground state energy for a given α is $\hbar\omega$ for $n = 1$, i.e., there is no zero point energy.

•It is also obvious from the analysis that the energy levels are created by the uniform gravitational field or the constant acceleration of the frame.

•Therefore if any transition takes place from some higher to lower energy levels, the emitted energy will not be of any kind of conventional or known type quanta. We call it as the dark quanta or Cosmic Phonon and the dark wave for its classical counter part. In the case of excitation to some higher energy levels the absorbed energy must also be in the form of cosmic phonon.

• Wien's displacement law:

$$\omega = \frac{\alpha}{c} \text{ hence } \lambda\alpha = \text{constant}$$

CMBR (Black body) case:

$$\lambda T = \text{constant}$$

for

$$\alpha \sim \frac{1}{x_l^2}$$

Then

$$\lambda \propto x_l^2$$

Effective refractive index μ:

$$\text{if } \mu \propto \alpha \text{ then } \mu\lambda = \text{constant}$$

If true for optical case

Change in wavelength of light during gravitational bending

Science

Science is the most forward outpost of human society, the scout of the future and the most reliable defender of the present.

Every science has two lifetimes. The first has to do with ideas, conceptions, laws and formulas. The second has to do with their translation into the hardware of technology– tools and instruments and machines.

No matter how abstract the gyration of the scientific mind, there is always a return to the real world of human being and to their needs.

Science perceives future problems before human practice gets to them. This fore-sight is not a favor of the gods or of geniuses, it is objective reality, at the heart of which lie the laws of development of society.

——————- ABC's of Quantum Mechanics, V. Rydnik, Mir