Schrödinger Equation in Uniformly Accelerated Frame

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- The Work is Based on the Principle of Equivalence. An accelerated frame of reference is equivalent to a frame at rest in presence of gavitational field of strength equal to the acceleration of the frame.

- The gravitational field is supposed to be produced by a strong gravitating object-say a black hole.

- A black hole is defined as a region of space-time that can not communicate with the external universe.

- The Event Horizon of a black hole is the boundary region or the surface of the black hole.

Rindler Coordinate

- The Rindler space-time coordinates are the just an uniformly accelerated frame transformation of the Minkowski metric of special relativity.

$$ct = \left(\frac{c^2}{g} + x'\right) \sinh\left(\frac{gt'}{c}\right) \quad \text{and} \quad x = \left(\frac{c^2}{g} + x'\right) \cosh\left(\frac{gt'}{c}\right)$$

- The inverse relations:

$$ct' = \frac{c^2}{2g} \ln\left(\frac{x + ct}{x - ct}\right) \quad \text{and} \quad x' = (x^2 - (ct)^2)^{1/2} - \frac{c^2}{g}$$

- The motion is assumed to be rectilinear and along x-direction, $dy' = dy$ and $dz' = dz$. the Rindler coordinates transform the Minkowski line element

$ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2$ to $ds^2 = \left(1 + \frac{gx'}{c^2}\right)^2 d(ct')^2 - dx'^2 - dy'^2 - dz'^2$

- The single particle classical Lagrangian:

- The action integral may be written as:

$$S = -\alpha_0 \int_a^b ds \equiv \int_a^b L dt$$

Hence for $\alpha_0 = m_0 c^2$

$$L = -m_0 c^2 \left[\left(1 + \frac{\alpha x}{c^2}\right)^2 - \frac{v^2}{c^2}\right]$$

- where $\alpha = g$ is the uniform acceleration of the particle along x-direction

- $v = u_x$, the particle velocity and m_0 is the rest mass of the particle.

- The three momentum vector of the particle can then be written as

$$\vec{p} = \frac{m_0 \vec{v}}{\left[\left(1 + \frac{\alpha x}{c^2}\right)^2 - \frac{v^2}{c^2}\right]^{1/2}}$$

- The Hamiltonian of the particle:

$$H = m_0 c^2 \left(1 + \frac{\alpha x}{c^2}\right) \left(1 + \frac{p^2}{m_0^2 c^2}\right)^{1/2}$$

- Quantum mechanical picture $\longrightarrow$ the classical dynamical variables x, $\vec{p}$ and H are treated as operators $\Longrightarrow$ the commutation relations

$$[x, p_x] = i\hbar \text{ and } [x, p_y] = [x, p_z] = 0$$

- The Schrödinger equation for the particle:

$$H\psi = \left(1 + \frac{\alpha x}{c^2}\right)\left(m_0 c^2 + \frac{p^2}{2m_0}\right)\psi = E\psi$$

Hence

$$\left(m_0 c^2 - \frac{\hbar^2}{2m_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)\psi = \frac{1}{1 + \frac{\alpha x}{c^2}} E\psi$$

- Since there is x-dependent term $\Longrightarrow$ the separable form of the solution:

$$\psi(x, y, z) = NX(x) \exp\left(-\frac{ip_y y}{\hbar}\right) \exp\left(-\frac{ip_z z}{\hbar}\right)$$

- On substituting $\psi(x, y, z)$:

$$\left(m_0 c^2 + \frac{p_y^2}{2m_0} + \frac{p_z^2}{2m_0}\right) X - \frac{\hbar^2}{2m_0}\frac{d^2 X}{dx^2} = \frac{1}{1 + \frac{\alpha x}{c^2}} EX$$

- The quantity within the first bracket on the left hand side is defined as $E_\perp$, the orthogonal part of the total energy, including the rest mass energy $\Longrightarrow$

$$E_\perp X - \frac{\hbar^2}{2m_0}\frac{d^2 X}{dx^2} = \frac{1}{1 + \frac{\alpha x}{c^2}} EX$$

- Introduce a new variable $u = 1 + \alpha x/c^2$ and define two constant quantities

$$a = \frac{2m_0c^4}{\hbar^2\alpha^2}E \quad \text{and} \quad b = \frac{8m_0c^4}{\hbar^2\alpha^2}E_\perp,$$

- Above equation takes the following form

$$\frac{d^2X}{du^2} + \frac{a}{u}X - \frac{b}{4}X = 0$$

- Defining $v = b^{1/2}u$ as a new variable and $\gamma = ab^{-1/2}$ as another constant quantity, $\Longrightarrow$

$$\frac{d^2X}{dv^2} + \left(-\frac{1}{4} + \frac{\gamma}{v}\right)X = 0$$

- On comparing this differential equation with that satisfied by the Whittaker function $M_{k,\mu}(x)$ given by

$$\frac{d^2}{dx^2}M_{k,\mu}(x) + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right)M_{k,\mu}(x) = 0$$

- we have $X(v) \equiv M_{k,\mu}(v)$, for $x = v$, $\mu = 1/2$ and $k = \gamma$. Then one can write down the solution in the explicit form as

$$X(v) = M_{\gamma,\frac{1}{2}}(v) = \exp\left(-\frac{v}{2}\right)vM(1-\gamma, 2, v)$$

•where $M(a,c,x) = {}_1F_1(a;c;x)$, the Confluent Hypergeometric function.

• From the knowledge of special functions, clearly, $M(1-\gamma,2,v)$ will be a polynomial and becomes zero for $v \longrightarrow \infty$, if the parameter $1-\gamma$ is 0 or a negative integer, i.e., $\gamma = n$, for $n = 1,2,3......$, the positive integers.

•Under such restricted situation, the solution can also be expressed in terms of Associated Laguerre function.

•This alternative form of wave function is then given by

$$X(v) = \exp\left(-\frac{v}{2}\right) v L^1_{\gamma-1}(v)$$

•The parameter γ is again have to be non-zero positive integer.

•It is exactly identical with one-dimensional hydrogen atom.

•The quantized form of energy eigen value of the particle in an uniformly accelerated frame is given by

$$E_n = n\hbar\frac{\alpha}{c} = n\hbar\omega \ \ \text{(say)}$$

where we have neglected p_y and p_z for a purely one-dimensional condition.

•Then accordingly $E_\perp = m_0c^2$, the rest mass energy of the particle.

•Surprisingly the differential equation satisfied by the particle is exactly identical with the equation for an one-dimensional hydrogen atom, whereas the quantized energy levels look like that of the energy levels for an one-dimensional quantum harmonic oscillator.

•The energy levels vary linearly with the quantum number n, instead of $1/n^2$, where the last one is the case for an one-dimensional hydrogen atom.

•The energy levels are observed to be independent of particle rest mass. It depends only on the acceleration α of the frame.

•It is obvious that $\omega \longrightarrow 0$ as $\alpha \longrightarrow 0$.

• Further, unlike the harmonic oscillator case, the minimum energy of the particle or the ground state energy for a given α is $\hbar\omega$ for $n = 1$, i.e., there is no zero point energy.

•It is also obvious from the analysis that the energy levels are created by the uniform gravitational field or the constant acceleration of the frame.

•Therefore if any transition takes place from some higher to lower energy levels, the emitted energy will not be of any kind of conventional or known type quanta. We call it as the dark quanta or Cosmic Phonon and the dark wave for its classical counter part. In the case of excitation to some higher energy levels the absorbed energy must also be in the form of cosmic phonon.

- Now we make the approximation

$$\frac{1}{1+\frac{\alpha x}{c^2}} \approx 1 - \frac{\alpha x}{c^2}$$

- Then we have with the separable solution

$$\frac{d^2 X}{dx^2} - \frac{2m_0 E\alpha}{\hbar^2 c^2} x X(x) = -\frac{2m_0}{\hbar^2}\left(E_k - \frac{p_\perp^2}{2m_0}\right) X(x)$$

where

$$\frac{p_\perp^2}{2m_0} = \frac{p_x^2 + p_y^2}{2m_0}$$

- is the orthogonal part of kinetic energy.

- Hence the parallel part of kinetic energy is given by

$$E_{||} = E_k - \frac{p_\perp^2}{2m_0}$$

- Let us put

$$\zeta = \left(\frac{2m_0 E\alpha}{\hbar^2 c^2}\right)^{1/3} x$$

- a new dimensionless variable and

$$E' = \frac{2m_0 E_{||}}{\hbar^2}\left(\frac{\hbar^2 c^2}{2m_0 E\alpha}\right)^{2/3}$$

- Then the above differential equation reduces to

$$\frac{d^2 X}{d\xi^2} + \xi X = 0$$

- with $\xi = E' - \zeta$.

- **This equation is of the same form as was obtained by Fowler and Nordheim in their original work on field emission of electrons**

- The mathematical reason behind the similar form of the differential equations is because of the same constant type driving fields for both the cases.

- In the case of Fowler-Nordheim emission, it is the constant attractive electrostatic field obtained from the potential of the form $C - Ex$, where C is the surface barrier, which is approximated with the work function of the metal and E is the uniform electrostatic field near the metal surface.

- The quantity $C - Ex$ acts as the driving potential for cold emission. Whereas in the case of particle production under strong gravitational force, the driving force is the uniform gravitational field say near the event horizon of the black hole.

- The complete form of solution is given by

$$\psi(x,y,z) = N \exp(-ip_y y/\hbar) \exp(-ip_z z/\hbar)(E' - \zeta)^{1/2} H^{(2)}_{1/3}\left[\frac{2}{3}(E' - \zeta)^{3/2}\right]$$

- With this solution, we have

$$\begin{aligned}\psi(x,y,z) &= N \exp\left(-i\frac{p_y y}{\hbar}\right) \exp\left(-\frac{ip_z z}{\hbar}\right) \\ &\quad (E' - \zeta)^{1/2} H_{1/3}\left[\frac{2}{3}(E' - \zeta)^{3/2}\right]\end{aligned}$$

- where N is the normalization constant.

- Since we expect oscillatory solution also along x-direction in the asymptotic region, we have replaced $J_{1/3}(x)$ by $H^{(2)}_{1/3}(x)$, the Hankel function of second kind.

- Now from the definitions

$$\xi = E' - \zeta = \frac{2m_0 E_{||}}{\hbar^2}\left(\frac{\hbar^2 c^2}{2m_0 E\alpha}\right) - \left(\frac{2m_0 E\alpha}{\hbar^2/c^2}\right)^{1/3} x,$$

• if it is assumed that for some local rest frame at a distance x_l from the centre of the black hole, in the asymptotic region, i.e., $x_l \gg$ the Schwarzschild radius, the gravitational field $\alpha = GM/x_l^2$, the quantity ξ as defined above can be expressed in terms of x_l in the following manner.

$$\xi \sim a x_l^{4/3} - b x_l^{1/3}$$

where a and b are real positive constants.

• The argument of the Hankel function is large enough and positive in this asymptotic region.

• The Hankel function can therefore be expressed as an oscillatory function in this uniformly accelerated frame at asymptotically large distance.

• On the other hand if it is assumed that the uniform acceleration for a local frame at x_l, close to the event horizon, is blue shifted, one can write

$$\alpha = \frac{GM}{x_l^2}\left[1 - \frac{R_s}{x_l}\right]^{-1/2}$$

• which gives the diverging value for α at the Schwarzschild radius, i.e. for $x_l = R_s = 2GM/c^2$.

• The value of ξ is negative near the event horizon and remain negative up to certain value of x for the local rest frames for which α's are quite large.

• To accommodate the negative values for ξ for a set of local rest frames, we replace ξ by $-\xi$, and then the modified form of Hankel function is given by

$$H_{1/3}^{(2)}\left(\exp\left(\frac{3}{2}\pi i\right)Q\right)$$

• Which may be expressed in terms of the modified Bessel function of first kind:

$$-\frac{1}{\sin(\pi/3)}\left[I_{-1/3}(Q)+\exp(i\pi/3)I_{1/3}(Q)\right]$$

• where $Q = 2\xi^{3/2}/3$.

• We define the particle density in the following manner in a particular local rest frame in presence of gravitational field α.

$$n = \text{constant} \mid \psi \mid^2$$

• The number density is large enough for the local rest frames near the event horizon where ξ's are negative.

• Follows from the expression for modified Bessel function of first kind for large Q:

$$I_\nu(Q) \sim \frac{1}{(2\pi Q)^{1/2}} \exp(Q)$$

• The physical reason for large particle number density near the event horizon is because of strong gravitational field, which produces more particles compared to far regions.

• This is also true in the case of Fowler-Nordheim field emission.

• More strong the electrostatic field more will be the electron emission rate.

• In this region the number density is given by:

$$n \sim \xi^{1/2} \exp(2Q)$$

• When ξ becomes positive, true for a frame quite far away from the event horizon, the wave function is given by the Hankel function.

• At $\xi = 0$, although the Hankel function diverges, the wave function vanishes in this particular frame of reference because of $\xi^{1/2}$ term.

• Same is true for the solution for $\xi < 0$, which matches exactly with $\xi > 0$ solution at $\xi = 0$.

• Further the Hankel function asymptotically becomes oscillatory in nature.

• The wave function for $\xi \longrightarrow \infty$ is given by

$$\psi(\xi) \sim \xi^{-1/4} \exp\left[-i\left(\xi - \frac{5\pi}{12}\right)\right]$$

• Hence in some local rest frame at x_l, which is far away from the event horizon, in presence of an uniform weak gravitational field

$$n(\xi \longrightarrow \infty) \sim (ax_l^{4/3} - bx_l^{1/3})^{-1/2}$$

• **We may divide the whole space out side the black hole into effectively five regions:**

• for the set of local rest frames in presence of uniform gravitational field, but far from the event horizon, the wave functions are oscillatory.

• For $\xi > 0$ but not large enough, the wave functions can be expressed in those frames in terms of Hankel function of second kind.

• At $\xi = 0$, the nature of the wave functions from both $\xi \longrightarrow 0+$ and $\xi \longrightarrow 0-$ show that it should vanish.

• For $\xi < 0$, but the magnitude is not large enough, the wave functions can be expressed in terms of modified Bessel function of first kind.

• Very close to the event horizon, where ξ is also less than zero but with very high in magnitude, the number density shows exponential growth and asymptotically diverges.

- In this work we have drawn some analogy of particle production near the event horizon of a black hole with that of field emission or cold emission of electrons from the metal surface in the non-relativistic scenario in a frame undergoing uniform accelerated motion in an otherwise flat space-time geometry.

- In the case of cold emission, the driving force is the strong external electrostatic field applied near the metal surface. The strong electrostatic field helps the electrons to tunnel out through the surface barrier. These electrons are liberated to the real world from the conduction band of the metal.

- Further in the case of cold emission, only electrons are liberated.

- Whereas for the black hole particle production, it is the strong gravitational field of the black hole near the event horizon is the driving force.

- Further, in the case of black hole emission the pairs come out from the quantum vacuum, where they are in the form of condensates.

- It is strongly believed that in the quantum field theoretic approach in curved space-time, the creation of particles at the event horizon is basically Schwinger type quantum tunneling process. In this article we have shown that in the non-relativistic approximation, it is also a tunneling process, but may be identified as gravitational Fowler-Nordheim emission. However, in the non-relativistic quantum mechanics the concept of vacuum does not exist.